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ASYZYGIES, MODULAR FORMS, AND THE SUPERSTRING MEASURE II *

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Abstract

Precise factorization constraints are formulated for the three-loop superstring chiral measure, in the separating degeneration limit. Several natural Ansätze in terms of polynomials in theta constants for the density of the measure are examined. None of these Ansätze turns out to satisfy the dual criteria of modular covariance of weight 6, and of tending to the desired degeneration limit. However, an Ansatz is found which does satisfies these criteria for the square of the density of the measure, raising the possibility that it is not the density of the measure, but its square which is a polynomial in theta constants. A key notion is that of totally asyzygous sextets of spin structures. It is argued that the Ansatz produces a vanishing cosmological constant.

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1 Introduction

Recently, the superstring measure to two loop order and for even spin structure was computed from first principles [1, 2, 3, 4, 5, 6, 7]. The construction relies on a careful treatment of supermoduli, chiral splitting and finite-dimensional gauge fixing determinants, and builds on earlier work in this direction [8, 9]. Although intermediate calculations are complex and intricate, the final form of the superstring measure turns out to be very simply expressed in terms of a new modular object, denoted by $\Xi_6[\delta](\Omega)$ in [4].

At present, no analogous derivation is available to 3-loop order and beyond. Some of the special simplicity of genus 2 does carry over to genus 3, in that no Schottky relations need to be imposed on the period matrix. The structure of supermoduli, however, becomes considerably more complex and, at present, the calculation appears formidable.

Therefore, the simplicity of the ultimate form of the two-loop superstring measure raises the question as to whether the genus 3 superstring measure might have a comparatively simple form in terms of natural modular objects. Constraints from holomorphicity, modular invariance, and physical factorization will provide powerful restrictions on any candidate measures. The precise form of the 2-loop measure gives a drastic constraint on the separating degeneration limits of the 3-loop measure.*

In this paper, we take a first step in the degeneration approach to the superstring measure by formulating a precise Ansatz for the 3-loop measure and verifying that it satisfies the correct factorization conditions when the worldsheet degenerates. Our Ansatz for the (chiral) superstring measure $d\mu[\Delta](\Omega^{(3)})$ can be described as follows. Set

$$d\mu[\Delta](\Omega^{(3)}) = \frac{\vartheta[\Delta](0,\Omega^{(3)})^4\,\Xi_6[\Delta](\Omega^{(3)})}{8\pi^4\,\Psi_9(\Omega^{(3)})}\prod_{I\leq J} d\Omega^{(3)}_{IJ} \tag{1.1}$$

Here Δ is a fixed even spin structure, $\Omega^{(3)} = \{\Omega^{(3)}_{IJ}\}$ is the period matrix of the genus 3 worldsheet, $\Psi_9(\Omega^{(3)}_{IJ})^2$ is the modular form $\Psi_{18}(\Omega^{(3)})$ of weight 18 constructed in [13], and the measure $\Psi_9(\Omega^{(3)}_{IJ})^{-1}\prod d\Omega_{IJ}$ has been shown to be holomorphic in [11]. The key term $\Xi_6[\Delta](\Omega^{(3)})$ is to be determined by the following constraints:

(i) $\Xi_6[\Delta](\Omega^{(3)})$ is holomorphic in $\Omega^{(3)}$ on the Siegel upper half space;

(ii) $\Xi_6[\Delta](\Omega^{(3)})$ is a modular covariant form of weight 6 in the sense that, under modular transformations sending $\Omega^{(3)}_{IJ} \to \tilde\Omega^{(3)}_{IJ} = (A\Omega^{(3)}+B)(C\Omega^{(3)}+D)^{-1}$, $\Delta \to \tilde\Delta$, we have

$$\Xi_6[\tilde\Delta](\tilde\Omega^{(3)}) = \epsilon(\Delta,M)^4 \det(C\Omega^{(3)}+D)^6\,\Xi_6[\Delta](\Omega^{(3)}), \tag{1.2}$$

*The constraints of modular invariance were used along these lines to guess the bosonic string measure to 2- and 3-loops in [10] and [11] respectively. A general theory based on constraints from modular invariance and physical factorization was developed in [12].

where $\epsilon(\Delta, M)$ is the same phase factor as in the modular transformation for ϑ-constants.

(*iii*) In the degeneration $t \to 0$, where the worldsheet separates into a genus 1 and a genus 2 surface of period matrices $\Omega^{(1)}$ and $\Omega^{(2)}$ respectively, we must have

$$\lim_{t\to 0} \Xi_6[\Delta](\Omega^{(3)}) = \eta(\Omega^{(1)})^{12}\, \Xi_6[\delta](\Omega^{(2)}), \tag{1.3}$$

where $\Xi_6[\delta](\Omega^{(2)})$ is the main new factor in the genus 2 superstring measure found in [1, 4].

The constraint (*iii*) on the degeneration limit of $\Xi_6[\Delta](\Omega^{(3)})$ is a consequence of the factorization properties of string amplitudes. To establish it, we require a precise formula for the degeneration limit of the measure $\Psi_9(\Omega^{(3)})^{-1} \prod d\Omega^{(3)}_{IJ}$, formula which is also one of the main results of this paper (see Theorem 1 below).

We should stress that the condition (*iii*) is very restrictive, since it applies to an *arbitrary* separating degeneration. Thus we have to expect $\Xi_6[\Delta](\Omega^{(3)})$ to be built of sums of many terms, different groups of which would tend to 0 in different limits.

The original expression for $\Xi_6[\delta](\Omega^{(2)})$ derived in [1, 2, 3, 4] depended very much on the fact that the worldsheet had genus 2. Since then, two alternate expressions have been found which can extend to higher genus [14]. A characterizing feature of these two expressions is that one of them is a sum over fourth powers of ϑ-constants of *triplets* of spin structures, while the other is a sum of second powers of ϑ-constants of *sextets* of spin structures. The key to determining which N-tuplets $\{\delta_i\}$ of spin structures should contribute to $\Xi_6[\delta](\Omega)$ turns out to be the notion of total asyzygies. Recall that to any triplet of spin structures $\{\delta_1, \delta_2, \delta_3\}$ is associated a modular invariant sign, namely the product

$$e(\delta_1, \delta_2, \delta_3) = \langle \delta_1 | \delta_2 \rangle \, \langle \delta_2 | \delta_3 \rangle \langle \delta_3 | \delta_1 \rangle \tag{1.4}$$

of relative signatures $\langle \delta | \epsilon \rangle = \exp 4\pi i (\delta' \epsilon'' - \epsilon' \delta'')$. A triplet of spin structures is said to be syzygous or asyzygous, depending on whether e is $+1$ or -1. The criteria for which triplets or sextets should contribute to $\Xi_6[\delta](\Omega)$ turns out to be entirely expressible in terms of asyzygies (see §5.1 below). Once the criteria for which triplets or sextets to include has been identified, one needs to find phase assinments $\epsilon(\delta; \{\delta_i\})$ with which to sum the contributions of various sextets. The phase assignments have to be consistent with modular invariance, which identifies them all up to a global phase.

These alternative descriptions of $\Xi_6[\delta](\Omega)$ suggest several possible generalizations to genus 3, all involving summations over monomials in $\vartheta[\Delta_i]$. They are listed in §5.2, where we describe also in detail their viability as Ansätze for the genus 3 superstring chiral measure $\Xi_6[\Delta](\Omega^{(3)})$. The net outcome is the following:

• A first Ansatz is in terms of sums of products of three fourth powers only, such as $\vartheta[\Delta_{i_1}]^4 \vartheta[\Delta_{i_2}]^4 \vartheta[\Delta_{i_3}]^4$. Using in particular the degeneration formulas of [14], we show

that none of this form exists which satisfies the criteria (*ii*) and (*iii*). More generally, the criterion (*iii*), requiring the appearance of $\eta(\Omega^{(1)})$, effectively prevents the rule for which N-tuplets to be included to remain the same for all genera.

• Next, we consider Ansätze involving sums of second powers, such as $\prod_{j=1}^{6} \vartheta[\Delta_{i_j}]^2$. In genus 2, the sextets which contribute to $\Xi_6[\delta](\Omega)$ can be characterized by the condition of δ-admissibility (see §5.1). This condition makes sense for all genera, but in genus 3, the set of such sextets (called Δ-admissible by extension) breaks up into many orbits under the subgroup of modular transformations fixing a given spin structure Δ. One particularly important orbit is the set of sextets which do not contain Δ, and which are *totally asyzygous*, in the sense that any of their sub-triplets is asyzygous. We refer to the other orbits as *partially asyzygous*. The partially asyzygous orbits do not appear to have as simple a description as the orbit of totally asyzygous sextets, although they can be identified by computer analysis.

The partially asyzygous orbits turn out not to be viable candidates for $\Xi_6[\Delta](\Omega^{(3)})$: computer analysis reveals that many of them do not admit consistent phase assignments $\epsilon(\Delta; \{\Delta_i\})$. Even when they do, their degeneration limits do not satisfy the criterion (*iii*) listed above. Thus we rule them out as Ansätze for $\Xi_6[\Delta](\Omega^{(3)})$.

We found the criterion of totally asyzygous sextets to be much more compelling: its key property is that the genus 2 sextets obtained by factorization from a totally asyzygous genus 3 sextet automatically satisfy the key condition of admissibility in genus 2 (see Lemma 1 in section §6.1). Furthermore, although these genus 2 sextets may be admissible but not δ-admissible, Lemma 2 in section §6.2 shows that the contributions of such sextets sum up to 0 if they are assigned phases consistent with modular invariance. Thus the Ansatz in terms of totally asyzygous sextets would satisfy the degenerating condition (*iii*) if phase assignments exist which are consistent with (*ii*). However, perhaps surprisingly, such a consistent phase assignment does not exist and (*ii*) cannot be satisfied. A simple example is provided in section §6.2.2.

• Another possible Ansatz could be in terms of sums of products of twelve first powers of ϑ, such as $\prod_{i=1}^{12} \vartheta[\Delta_i](0, \Omega^{(3)})$. The criterion for which dozens $\{\Delta_i\}_{1 \leq i \leq 12}$ to include is difficult to guess from the genus 2 case. There is no consistent phase assignments if the dozens are assumed to consist of a pair of totally asyzygous sextets, and more generally, no consistent sign assignments appear possible.

Thus, we are led to believe that no candidate for $\Xi_6[\Delta](\Omega^{(3)})$ exists which is a polynomial in ϑ. On the other hand, consistent modular covariant assignments $\epsilon(\Delta; \{\Delta_i\}, \{\Delta'_i\})$ do exist for suitable bilinear combinations of pairs of totally asyzygous sextets of $\vartheta[\Delta_i]^2$. This suggests that only $\Xi_6[\Delta](\Omega^{(3)})^2$ is a polynomial in ϑ-constants. We find that, for a suitable integer normalization factor N, and a suitable choice of multiplicities N_{pq} of the

orbits $\mathcal{Q}_{pq}$ of pairs $\{\Delta_i, \Delta_i'\}$ of totally asyzygous sextets under the subgroup of $Sp(6, \mathbf{Z})$ fixing Δ, the expression

$$\Xi_6[\Delta](\Omega^{(3)})^2 = \frac{1}{2^8 N} \sum_{pq} N_{pq} \sum_{(\{\Delta_i\},\{\Delta_i'\}) \in \mathcal{Q}_{pq}} \epsilon(\Delta; \{\Delta_i\}, \{\Delta_i'\}) \prod_{i=1}^{6} \vartheta^2[\Delta_i] \prod_{i=1}^{6} \vartheta^2[\Delta_i'] \tag{1.5}$$

does satisfy all the conditions implied by (*i*)-(*iii*) for the square of $\Xi_6[\Delta](\Omega^{(3)})$, for arbitrary separating degeneration limits. In particular, it is a highly non-trivial result that in any separating degeneration limit of this form to a genus 2 and a genus 1 surface, the limit becomes a perfect square. In general, these expressions will not admit holomorphic square roots away from the separating degeneration limit. If there exists a specific choice of multiplicities N_{pq} (not all 0) which guarantees the existence of a holomorphic square root, then (1.5) will single out a compelling candidate for the genus 3 superstring measure. The existence of such a holomorphic square root is known to occur at genus 3 in at least one other instance, namely the modular form $\Psi_9(\Omega^{(3)}) = \prod_\Delta \vartheta[\Delta](0, \Omega^{(3)})^{\frac{1}{2}}$, which is known to be the (unexpectedly) holomorphic square root of $\Psi_{18}(\Omega^{(3)})$.

The remainder of this paper is organized as follows. In section 2, the general criterion for physical factorization is spelled out for the superstring measure. In section 3, the factorization properties of the bosonic factors in the genus 3 measure are derived. In section 4, the construction of the genus 3 superstring measure is formulated as a degeneration problem. In section 5, the consistency with criteria *(i), (ii), (iii)* above of various candidates is analyzed and (1.5) is constructed.

2 Factorization of the superstring measure

The main goal of this section is to derive the precise degeneration constraints which the 3-loop superstring measure must satisfy when a separating cycle in the worldsheet $\Sigma^{(3)}$ is pinched to a point, and $\Sigma^{(3)}$ separates into a torus $\Sigma^{(1)}$ and a genus 2 surface $\Sigma^{(2)}$.

2.1 Geometric picture of factorization

We begin with the geometric description of the moduli space of Riemann surfaces near the divisor of surfaces with nodes, as provided by the following well-known construction [15].

Let $\Sigma^{(1)}$ and $\Sigma^{(2)}$ be two Riemann surfaces of genus h_1 and h_2, let $p_1 \in \Sigma^{(1)}$, $p_2 \in \Sigma^{(2)}$ be two given points, and let $|z_1| < 1$, $|z_2| < 1$ be local coordinates on $\Sigma^{(1)}$ and $\Sigma^{(2)}$ which are centered at p_1 and p_2 respectively. Let $\mathcal{S}$ be the surface given by $\mathcal{S} = \{(X, Y, t);\ XY = t\ , |X| < 1, |Y| < 1, |t| < 1\}$, and construct the fibration $\mathcal{C}$ of surfaces over the unit disk $\{t; |t| < 1\}$ given by

$$\mathcal{C} = \{(z_1, t); z_1 \in \Sigma^{(1)},\ |z_1| > |t|\} \ \cup\ \mathcal{S}\ \cup\ \{(z_2, t); z_1 \in \Sigma^{(2)},\ |z_2| > |t|\}, \tag{2.1}$$

with the following identifications

$$\begin{aligned} (z_1, t) &\sim (z_1, \frac{t}{z_1}, t) \ \ \text{for } z_1 \in \Sigma^{(1)},\ |t| < |z_1| < 1 \\ (z_2, t) &\sim (\frac{t}{z_2}, z_2, t) \ \ \text{for } z_2 \in \Sigma^{(2)},\ |t| < |z_2| < 1. \end{aligned} \tag{2.2}$$

For each $t \neq 0$, the fiber of $\mathcal{S}$ above t can be identified with the annulus $A_t = \{X; |t| < |X| < 1\}$. Thus the fiber of $\mathcal{C}$ above t is a regular surface Σ_t of genus $h = h_1 + h_2$, which can be covered by the three overlapping charts $\Sigma^{(1)} \setminus \{|z_1| > |t|\}$, A_t, and $\Sigma^{(2)} \setminus \{|z_2| > |t|\}$, with the identifications

$$z_1 \ \sim\ X\ \sim\ \frac{t}{z_2}, \quad \text{for } \ |t| < |z_1|, |z_2| < 1. \tag{2.3}$$

2.2 Physical picture of factorization

In the physical picture, we view the surface Σ_t rather as the disjoint union

$$\Sigma_t = \Sigma^{(1)}_{in}\ \cup\ A_t\ \cup \Sigma^{(2)}_{out} \tag{2.4}$$

where we have set $\Sigma^{(1)}_{in} = \Sigma^{(1)} \setminus \{|z_1| < 1\}$, and $\Sigma^{(2)}_{out} = \Sigma^{(1)} \setminus \{|z_1| < 1\}$. In a given conformal field theory, the surfaces with boundary $\Sigma^{(1)}_{in}$, $\Sigma^{(2)}_{out}$ define two states $\langle \Sigma^{(1)}_{in}|$ and

$|\Sigma^{(2)}_{out}\rangle$. To make contact with the Hamiltonian picture, we can use the exponential map $\xi \to X = t^{\frac{1}{2}} e^{\xi}$ to identify the annulus A_t with a cylinder

$$\{\xi = \xi_0 + i\xi_1;\ 0 \le \xi_1 \le 2\pi,\ -\frac{1}{2}\ln\frac{1}{|t|} < \xi_0 < \frac{1}{2}\ln\frac{1}{|t|}\}. \tag{2.5}$$

Now the operators for time and space translations are the Hamiltonian $H = L_0 + \bar{L}_0$ and the momentum operator $P = L_0 - \bar{L}_0$ †. If we view ξ_0 as "time", and ξ_1 as "space", then the shift in time and the shift in space corresponding to the cylinder are given respectively by the length of the cylinder and the phase shift in ξ as the point X moves on a straight line from $X = |t|$ to $X = 1$. This gives $-\ln|t|$ for the shift in time and $\arg(t)$ for the shift in space, since $\xi = |t|^{\frac{1}{2}} e^{-i\frac{1}{2}\arg(t)}$ and $\xi = |t|^{-\frac{1}{2}} e^{i\frac{1}{2}\arg(t)}$ are the points on the cylinder corresponding to $X = |t|^{\frac{1}{2}}$ and $X = 1$. The cylinder corresponds then to the following operator insertion

$$\exp\Big(i\arg(t)(L_0 - \bar{L}_0)\Big)\ \exp\Big(\ln(|t|)(L_0 + \bar{L}_0)\Big) = t^{L_0}\,\bar{t}^{\bar{L}_0} \tag{2.6}$$

and hence the partition function $\mathcal{Z}_t$ corresponding to the surface Σ_t is given by

$$\mathcal{Z}_t = \langle \Sigma^{(1)}_{in}|\ t^{L_0}\,\bar{t}^{\bar{L}_0}\ |\Sigma^{(2)}_{out}\rangle \tag{2.7}$$

To obtain the degenerating limit $t \to 0$, we insert a basis of states $|\psi_\alpha\rangle$ diagonalizing $t^{L_0}\,\bar{t}^{\bar{L}_0}$

$$\mathcal{Z}_t = \sum_\alpha \langle \Sigma^{(1)}_{in}|\psi_\alpha\rangle\,\langle\psi_\alpha|\ t^{L_0}\,\bar{t}^{\bar{L}_0}\ |\psi_\alpha\rangle\,\langle\psi_\alpha|\Sigma^{(2)}_{out}\rangle \tag{2.8}$$

The descendant states $|\psi_\alpha\rangle$ contribute lower order terms in the limit $t \to 0$. To identify the leading contribution, we need thus to consider only primary states. In the case of string propagation, before the GSO projection, the state with lowest m^2 is the tachyon with $m^2 = -2$. By momentum conservation, its momentum must be $k^\mu = 0$ (it is not on-shell, but intermediate states do not have to be on-shell). Since the vertex for tachyon emission with momentum 0 is just the identity, the leading term for $\mathcal{Z}_t$ is given by

$$\mathcal{Z}_t = \mathcal{Z}^{(1)} \cdot t^{-2}\bar{t}^{-2} \cdot \mathcal{Z}^{(2)} + O(|t|^{-3}) \tag{2.9}$$

where $\mathcal{Z}^{(1)}$ and $\mathcal{Z}^{(2)}$ are the partition functions for the surfaces $\Sigma^{(1)}$ and $\Sigma^{(2)}$.

To deal with spin structures, we start from surfaces $\Sigma^{(i)}$ with canonical homology bases $A^{(i)}_I$, $B^{(i)}_I$, $\#(A^{(i)}_I \cap B^{(i)}_J) = \delta_{IJ}$, $\#(A^{(i)}_I \cap A^{(i)}_J) = 0$, $\#(B^{(i)}_I \cap B^{(i)}_J) = 0$ for $1 \le I, J \le h_i$.

† Since all conformal anomalies ultimately cancel, we can ignore the contribution of the central charge when we map the annulus into the cylinder.

Then the combined bases give a canonical basis for the genus $h_1 + h_2$ surface Σ_t. With this choice of homology bases, a spin structure Δ can be identified with an assignment of either 0 or 1/2 to each homology cycle of Σ_t, and hence with a pair (δ_1, δ_2), with δ_i a spin structure on the surface $\Sigma^{(i)}$

$$\Delta = \begin{pmatrix} \delta_2 \\ \delta_1 \end{pmatrix}. \tag{2.10}$$

In a conformal field theory where the fields are world sheet fermions requiring a spin structure, the preceding degeneration formula becomes

$$\mathcal{Z}_t[\Delta] = \mathcal{Z}^{(1)}[\delta_1] \cdot t^{-2}\bar{t}^{-2} \cdot \mathcal{Z}^{(2)}[\delta_2] + \emptyset(|t|^{-3}). \tag{2.11}$$

2.3 Factorization of the genus 3 superstring measure

We formulate now the precise degeneration constraint for the superstring measure when the worldsheet $\Sigma = \Sigma_t$ is of genus $h = 3$ and degenerates into two surfaces $\Sigma^{(1)}$ and $\Sigma^{(2)}$ of genus $h_1 = 1$ and $h_2 = 2$.

We shall assume that, at loop order h, the vacuum-to-vacuum superstring amplitude is of the form

$$\mathcal{A} = \sum_{\Delta, \bar{\Delta}} c_{\Delta,\bar{\Delta}} \int_{\mathcal{M}_h} (\det \operatorname{Im} \Omega^{(h)})^{-5} \; d\mu[\Delta](\Omega^{(h)}) \wedge \overline{d\mu[\bar{\Delta}](\Omega^{(h)})} \tag{2.12}$$

where $c_{\Delta,\bar{\Delta}}$ are suitable phases, and the sum over the spin structures $\Delta, \bar{\Delta}$ corresponds to the GSO projection, which projects out the tachyon and produces space-time supersymmetry. The space $\mathcal{M}_h$ is the moduli space of Riemann surfaces of genus h. We always fix a homology basis, and view each Riemann surface as characterized by its period matrix $\Omega^{(h)} = \{\Omega^{(h)}_{IJ}\}_{1 \leq I,J \leq h}$. The form $d\mu[\Delta](\Omega)$ is a $(3h-3, 0)$ holomorphic form on $\mathcal{M}_h$, transforming under modular transformations in such a way that the full expression above is modular invariant. It is called the (chiral) superstring measure at genus h.

Near $t = 0$, the $3h - 3$ moduli parametrizing Σ_t can be chosen to be the $3h_1 - 3$ and $3h_2 - 3$ moduli for the surfaces $\Sigma^{(1)}$ and $\Sigma^{(2)}$, together with the 3 parameters p_1, p_2 and t. The degeneration formulas derived above for conformal field theory suggest imposing the following degeneration constraint for the chiral superstring measure

$$d\mu[\Delta](\Omega) = d\mu[\delta_1](\Omega^{(1)}) \wedge \frac{dt}{t^2} \wedge d\mu[\delta_2](\Omega^{(2)}) \wedge dp_1 \wedge dp_2 + \emptyset(t^{-1}) \tag{2.13}$$

As usual, these formulas hold for $h_1, h_2 \geq 2$. When $h_1 = 1$, the counting is slightly different, since p_1 and its differential are no longer relevant due to translation invariance on the torus. This is actually the case of main interest in the present paper, so we make

the above formula more explicit in this case: the moduli for $\Sigma^{(1)}$ is then a single parameter $\Omega^{(1)}$, and the superstring measure for one-loop is $\vartheta^4[\delta_1](\Omega^{(1)})/2^5\pi^4\eta^{12}(\Omega^{(1)})$ (see e.g. [4], eq. (8.2)). Thus the degeneration constraint for the chiral superstring measure at genus h when the worldsheet separates into a torus $\Sigma^{(1)}$ and a genus $h-1$ surface $\Sigma^{(2)}$ is given by

$$d\mu[\Delta](\Omega) = \frac{\vartheta^4[\delta_1](\Omega^{(1)})}{2^5\pi^4\eta^{12}(\Omega^{(1)})}\, d\Omega^{(1)} \wedge \frac{dt}{t^2} \wedge d\mu[\delta_2](\Omega^{(2)}) \wedge dp_2 + \mathcal{O}(t^{-1}). \tag{2.14}$$

2.4 Factorization of the genus 3 bosonic string measure

Although this paper is mainly concerned with the genus 3 superstring measure and its degeneration limit, we take the opportunity to discuss also similar issues for the bosonic string, partly as a check later on our method. The measure for the bosonic string in the critical dimension is of the form

$$\mathcal{A} = \int_{\mathcal{M}_h} (\det \mathrm{Im}\,\Omega)^{-13}\, d\mu_B(\Omega) \wedge \overline{d\mu_B(\Omega)} \tag{2.15}$$

where $d\mu_B(\Omega)$ is holomorphic. Because the intermediate state of lowest mass is still the tachyon, the measure $d\mu_B(\Omega)$ satisfies the same degeneration constraint as in (2.13). When the worldsheet Σ degenerates into a torus $\Sigma^{(1)}$ and a surface of genus 2, the degeneration constraint can be written as

$$d\mu_B(\Omega) = \frac{d\Omega^{(1)}}{(2\pi)^{12}\eta^{24}(\Omega^{(1)})} \wedge \frac{dt}{t^2} \wedge d\mu_B(\Omega^{(2)}) \wedge dp_2 + \mathcal{O}(t^{-1}) \tag{2.16}$$

where $(2\pi)^{-12}\eta^{-24}(\Omega^{(1)})\, d\Omega^{(1)}$ is the genus 1 bosonic string measure, with the conventions of [8] and the normalization $d^2\Omega^{(1)}/(8\pi^2\,\mathrm{Im}\,\Omega^{(1)})^2$ for the $SL(2,\mathbf{R})$ invariant measure on the Siegel upper half space.

3 The measure $\Pi_{I\leq J}\, d\Omega^{(3)}_{IJ}/\Psi_9(\Omega^{(3)})$ in genus 3

An important feature of the chiral superstring measure $d\mu[\Delta](\Omega^{(h)})$ is that it is a holomorphic $(3h-3,0)$ form. To find it, we begin by constructing a natural holomorphic $(3h-3,0)$ form $d\mu_B(\Omega^{(h)})$ on $\mathcal{M}_h$ (later identified with the chiral bosonic measure, but this is not essential for our considerations), so that the problem of finding $d\mu[\Delta](\Omega^{(h)})$ reduces to that of finding the density $d\mu[\Delta]/d\mu_B$. In genera $h=2$ and $h=3$, we can exploit the fact that $\mathcal{M}_h$ and the Siegel upper half space of symmetric matrices with positive imaginary part have the same dimension, and henceforth we consider only these cases.

3.1 The modular forms $\Psi_{18}(\Omega^{(3)})$ and $\Psi_{10}(\Omega^{(2)})$

Recall that on a surface Σ of genus h, there are 2^{2h} spin structures, of which $2^{h-1}(2^h+1)$ are even and $2^{h-1}(2^h-1)$ are odd. The parity of a spin structure Δ corresponds to the parity in ζ of the ϑ-function $\vartheta[\Delta](\zeta,\Omega^{(h)})$, which is also the parity of the number of independent holomorphic spinors of spin structure Δ. The properties of ϑ-functions which we need can be found in [4], §2.1-§2.3 and [7], Appendix B. For convenience, we restate here the transformations of spin structures $\Delta\to\tilde\Delta$ and ϑ-constants $\vartheta[\Delta](0,\Omega^{(h)})\to\vartheta[\tilde\Delta](0,\tilde\Omega^{(h)})$ under modular transformations

$$\tilde\Omega^{(h)} = (A\Omega^{(h)}+B)(C\Omega^{(h)}+D)^{-1}, \qquad M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in Sp(2h,\mathbf{Z}). \tag{3.1}$$

If we write $\Delta=(\Delta'|\Delta'')$ and $\tilde\Delta=(\tilde\Delta'|\tilde\Delta'')$, they are given by

$$\begin{pmatrix}\tilde\Delta'\\ \tilde\Delta''\end{pmatrix} = \begin{pmatrix} D & -C\\ -B & A\end{pmatrix}\begin{pmatrix}\Delta'\\ \Delta''\end{pmatrix} + \frac{1}{2}\mathrm{diag}\begin{pmatrix} CD^T\\ AB^T\end{pmatrix} \tag{3.2}$$

and by

$$\vartheta[\tilde\Delta](0,\tilde\Omega) = \epsilon(\Delta,M)\det(C\Omega^{(h)}+D)^{\frac12}\,\vartheta[\Delta](0,\Omega^{(h)}), \tag{3.3}$$

where $\epsilon(\Delta,M)$ is an eighth root of unity, which depends on both the spin structure Δ and the modular transformation M. There is no simple closed formula for $\epsilon(\Delta,M)$, but its values for $h=2$ on generators of $Sp(4,\mathbf{Z})$ can be found in [4], §2.3.

The above transformation for ϑ-constants should be compared with the defining transformation law for modular forms $\Phi(\Omega)$ of a given weight w

$$\Phi(\tilde\Omega^{(h)}) = \det(C\Omega^{(h)}+D)^w\,\Phi(\Omega^{(h)}) \tag{3.4}$$

which do not involve roots of unity such as $\epsilon(\Delta,M)$. Nevertheless, the following natural form can be defined using the even ϑ-constants

$$\Psi_{2^{h-1}(2^h+1)k}(\Omega^{(h)}) = \prod_{\Delta\ even}\vartheta^{2k}[\Delta](0,\Omega^{(h)}) \tag{3.5}$$

It has been shown by Igusa [13] that in genus $h = 2$ and $h = 3$, $\Psi_{2^{h-1}(2^h+1)k}(\Omega^{(h)})$ are modular forms of weight $2^{h-1}(2^h + 1)k$ when $k = 1$ and $k = 1/2$ respectively.

Let these forms be denoted by $\Psi_{10}(\Omega^{(2)})$ and $\Psi_{18}(\Omega^{(3)})$ respectively. It is well-known that the form $\Psi_{10}(\Omega^{(2)})$ has no zero inside the moduli space of Riemann surfaces of genus 2, while the form $\Psi_{18}(\Omega^{(3)})$ vanishes exactly of second order along the variety of hyperelliptic surfaces of genus 3 [11]. Indeed, $\Psi_{2^{h-1}(2^h+1)}(\Omega^{(h)})$ vanishes if and only if a ϑ-constant vanishes for some even spin structure Δ. Since the parity of the number of independent holomorphic spinors is the same as the parity of Δ, this means that there are at least 2 independent holomorphic spinors of spin structure Δ. By the Riemann-Roch theorem, the number of zeroes of a holomorphic spinor is always $(h - 1)$. In genus $h = 2$, a holomorphic spinor has then exactly one zero, and the ratio of two linearly independent holomorphic spinors would be a meromorphic function with exactly one zero and one pole. Such a function provides a one-to-one correspondence between the given Riemann surface and the sphere, contradicting our initial assumption that $h = 2$. Similarly, when $h = 3$, a holomorphic spinor has 2 zeroes, and the ratio of two linearly independent holomorphic spinors is a meromorphic function with two zeroes and two poles. Such a function provides a two-to-one correspondence with the sphere, and thus the Riemann surface must be hyperelliptic. Conversely, if $s^2 = \prod_{i=1}^{8}(x - u_i)$ is a hyperelliptic surface of genus 3, then $s^{-\frac{1}{2}}(dx)^{\frac{1}{2}}$ and $xs^{-\frac{1}{2}}(dx)^{\frac{1}{2}}$ define two holomorphic spinors associated with an even spin structure. Thus $\Psi_{18}(\Omega^{(3)})$ vanishes at such surfaces (in fact, to second order), and the proof of the claim is complete.

Since the form $\Psi_{18}(\Omega^{(3)})$ vanishes of second order, we can follow [11] and obtain a holomorphic character $\Psi_9(\Omega^{(3)})$ by taking its square root

$$\Psi_9(\Omega^{(3)})^2 = \Psi_{18}(\Omega^{(3)}) \tag{3.6}$$

In genus $h = 2$ and $h = 3$, the moduli space $\mathcal{M}_h$ and the Siegel upper half space have the same dimension, which is 3 and 6 respectively. An integral over $\mathcal{M}_h$ can be identified with an integral over a fundamental domain of the modular group $Sp(2h, \mathbf{Z})$ in the Siegel upper half space. On this space, we can introduce the following holomorphic $(3h - 3, 0)$ forms [‡]

$$\begin{aligned} &\frac{1}{\Psi_{10}(\Omega^{(2)})} \prod_{1 \le I \le J \le 2} d\Omega_{IJ}^{(2)}, \quad \text{for genus } h = 2 \\ &\frac{1}{\Psi_9(\Omega^{(3)})} \prod_{1 \le I \le J \le 3} d\Omega_{IJ}^{(3)}, \quad \text{for genus } h = 3. \end{aligned} \tag{3.7}$$

[‡]The ordering of the forms $d\Omega_{IJ}^{(h)}$ in these measures is a matter of convention. We shall ignore the resulting $\pm$ signs and sometimes denote the resulting volume form just by $d^{h(h+1)/2}\Omega^{(h)}$.

Both measures are holomorphic on the Siegel upper half space. This is obvious when $h = 2$. When $h = 3$, this is due to [11], who showed that the form $\prod_{I \le J} d\Omega^{(3)}_{IJ}$ also vanishes along the variety of hyperelliptic surfaces, so that the zeroes in the denominator $\Psi_9(\Omega^{(3)})$ are cancelled by the measure factor.

It follows from Igusa's classification theorem for genus 2 modular forms that the bosonic string measure is actually given in genus $h = 2$ by [11, 10]

$$d\mu_B(\Omega^{(2)}) = \frac{c_2}{\Psi_{10}(\Omega^{(2)})} \prod_{1 \le I \le J \le 2} d\Omega^{(2)}_{IJ}, \tag{3.8}$$

where c_2 is an overall constant. This constant was in fact evaluated in [4] §7.1, and was found to be $c_2 = \pi^{-12}$. There is no such classification theorem in genus 3 or higher, but cogent arguments have been proposed for the similar relation in genus 3 to hold [11]

$$d\mu_B(\Omega^{(3)}) = \frac{c_3}{\Psi_9(\Omega^{(3)})} \prod_{1 \le I \le J \le 3} d\Omega^{(3)}_{IJ} \tag{3.9}$$

with c_3 another overall constant. As part of our program for determining the genus 3 superstring measure, we shall present further evidence for this relation below.

3.2 Degeneration of $\Psi_9^{-1}(\Omega^{(3)}) \prod_{I \le J} d\Omega^{(3)}_{IJ}$

The superstring chiral measure will be identified by its density with respect to the basic measure $\Psi_9(\Omega^{(3)}) \prod_{I \le J} d\Omega^{(3)}_{IJ}$. In order to reformulate the degeneration constraints (2.13) for the superstring measure in terms of degeneration constraints for its density, we need the precise degeneration limit of the measure $\Psi_9(\Omega^{(3)}) \prod_{I \le J} d\Omega^{(3)}_{IJ}$. This is given in the following theorem:

Theorem 1 *In the degeneration limit given by §2.1, $t \to 0$, we have*

$$\frac{1}{\Psi_9(\Omega^{(3)})} \prod_{1 \le I \le J \le 3} d\Omega^{(3)}_{IJ} = \frac{1}{(2\pi)^6} \frac{\prod_{I \le J} \Omega^{(2)}_{IJ}}{\Psi_{10}(\Omega^{(2)})} \wedge \frac{d\Omega^{(1)}}{\eta(\Omega^{(1)})^{24}} \wedge \frac{dt}{t^2} \wedge dp_2 + \emptyset(\frac{1}{t}). \tag{3.10}$$

Proof. We consider the parametrization of surfaces Σ_t degenerating into two surfaces $\Sigma^{(1)}$ and $\Sigma^{(2)}$ described in §2.1. As indicated there, we choose canonical homology bases $(A^{(i)}_{I_i}, B^{(i)}_{I_i})$, so that the union of these cycles constitutes a canonical homology basis for Σ. Let $(\omega^t_{I_1}, \omega^t_{I_2})$ be the basis of holomorphic Abelian differentials on Σ_t dual to the (A_{I_1}, A_{I_2}) cycles. Then these holomorphic differentials have the following asymptotic behavior as

$t \to 0$ [15]

$$
\begin{aligned}
\omega^t_{I_1}(z) &= \begin{cases} \omega_{I_1}(z) + \frac{t}{4}\omega_{I_1}(p_1)\omega^{(1)}_{p_1}(z) + \O(t^2) & \text{when} \quad z \in \Sigma^{(1)} \\ \frac{t}{4}\omega_{I_1}(p_1)\omega^{(2)}_{p_2}(z) + \O(t^2) & \text{when} \quad z \in \Sigma^{(2)} \end{cases} \\
\omega^t_{I_2}(z) &= \begin{cases} \frac{t}{4}\omega_{I_2}(p_2)\omega^{(1)}_{p_1}(z) + \O(t^2) & \text{when} \quad z \in \Sigma^{(1)} \\ \omega_{I_2}(z) + \frac{t}{4}\omega_{I_2}(p_2)\omega^{(2)}_{p_2}(z) + \O(t^2) & \text{when} \quad z \in \Sigma^{(2)} \end{cases}
\end{aligned}
\tag{3.11}
$$

Here, $\omega^{(i)}_{p_a}$ refers to the meromorphic differential on surface $\Sigma^{(i)}$, $i = 1, 2$ with a double pole at p_i, while ω_{I_i} refers to a basis of holomorphic differentials on surface $\Sigma^{(i)}$.

The components of the period matrix behave as follows [15],

$$
\begin{aligned}
\Omega^t_{I_1J_1} &= \Omega^{(1)}_{I_1J_1} + \frac{i\pi}{2}t\omega_{I_1}(p_1)\omega_{J_1}(p_1) & \qquad \Omega^t_{I_1J_2} &= \frac{i\pi}{2}t\omega_{I_1}(p_1)\omega_{J_2}(p_2) \\
\Omega^t_{I_2J_2} &= \Omega^{(2)}_{I_2J_2} + \frac{i\pi}{2}t\omega_{I_2}(p_2)\omega_{J_2}(p_2) & \qquad \Omega^t_{I_2J_1} &= \frac{i\pi}{2}t\omega_{I_2}(p_2)\omega_{J_1}(p_1)
\end{aligned}
\tag{3.12}
$$

where $\Omega^{(i)}$ refers to the period matrix on the surface $\Sigma^{(i)}$.

Henceforth, we consider the case where $h_1 = 1$ and $h_2 = 2$. It is convenient to set $\Omega^{(1)} = \tau$, $\Omega^{(2)} = \Omega$, and use the following notations,

$$
\Omega^{(3)} = \begin{pmatrix} \Omega_{11} & \Omega_{12} & \tau_1 \\ \Omega_{12} & \Omega_{22} & \tau_2 \\ \tau_1 & \tau_2 & \tau_3 \end{pmatrix} \qquad \qquad \begin{cases} \tau_1 &= \frac{i\pi}{2}t\ \omega_1(p_2)\omega_0(p_1) \\ \tau_2 &= \frac{i\pi}{2}t\ \omega_2(p_2)\omega_0(p_1) \end{cases}
\tag{3.13}
$$

Here $\omega_I(p_2)$ denote the genus 2 holomorphic differentials and ω_0 denotes the genus 1 holomorphic differential which is just the constant 1 in the usual parametrization of the torus of modulus τ as $\mathbf{C}/\mathbf{Z} + \tau\mathbf{Z}$, since the homology basis $(A^{(1)}_1, B^{(1)}_1)$ has been fixed.

The ϑ-constants at genus 3 for even spin structures behave differently in the separating limit depending on whether the spin structures on the genus 2 and genus 1 components are both even or both odd. We have the following limits,

$$
\begin{aligned}
\vartheta\begin{bmatrix} \delta \\ \mu \end{bmatrix}(0, \Omega^{(3)}) &= \vartheta[\delta](0, \Omega)\ \vartheta[\mu](0, \tau) + \O(t) \\
\vartheta\begin{bmatrix} \nu \\ \nu_0 \end{bmatrix}(0, \Omega^{(3)}) &= \frac{t}{4}\omega_0(p_1)\vartheta'_1(0, \tau)h_\nu(p_2)^2 + \O(t^2)
\end{aligned}
\tag{3.14}
$$

Here, δ (resp. ν) denote an even (resp. odd) genus 2 spin structure, while μ denotes an even genus 1 spin structure and ν_0 denotes the unique genus 1 odd spin structure. Furthermore, we use the familiar notation,

$$
h_\nu(z)^2 \equiv \omega_I(z)\partial^I\vartheta[\nu](0, \Omega) \tag{3.15}
$$

This square is defined for any surface, while its square root, h_ν in single-valued only on a surface with spin structure ν.

a) The limit of Ψ_{18}

We are now in a position to study the limit of the modular form Ψ_{18} and its square root Ψ_9. In genus 3, there are 36 even spin structures, of which 30 separate into two even spin structures in genus 1 and 2, and 6 separate into two odd spin structures in genus 1 and 2. In the first group of 30, the spin structures obtained after degeneration run over all 10 genus 2 even spin structures and over all 3 genus 1 even spin structures. Similarly, in the second group of 6, the spin structures obtained after degeneration run over all 6 genus 2 even spin structures. Thus we obtain

$$\Psi_{18}(\Omega^{(3)}) = \prod_{\delta,\mu} \left(\vartheta[\delta](0,\Omega)\vartheta[\mu](0,\tau)\right) \ \prod_\nu \left(\frac{t}{4} h_\nu(p_2)^2 \omega_0(p_1)\vartheta_1'(0,\tau)\right) + \emptyset(t^7) \tag{3.16}$$

In view of the well-known genus 1 identities,

$$\vartheta_1'(0,\tau) = -2\pi\eta(\tau)^3, \qquad \prod_\mu \vartheta[\mu](0,\tau) = 2\eta(\tau)^3, \tag{3.17}$$

and the definition of $\Psi_{10}(\Omega)$, this can be rewritten as

$$\begin{aligned} \Psi_{18}(\Omega^{(3)}) &= \Psi_{10}(\Omega)^{3/2} \left(2\eta(\tau)^3\right)^{10} \left(\frac{t}{4}\right)^6 \omega_0(p_1)^6 \left(-2\pi\eta(\tau)^3\right)^6 \prod_\nu h_\nu(p_2)^2 + \emptyset(t^7) \\ &= 2^4\pi^6 t^6 \omega_0(p_1)^6 \Psi_{10}(\Omega)^{3/2} \eta(\tau)^{48} \prod_\nu h_\nu(p_2)^2 + \emptyset(t^7) \end{aligned} \tag{3.18}$$

Taking the square root, we find

$$\Psi_9(\Omega^{(3)}) = 4\pi^3 t^3 \omega_0(p_1)^3 \Psi_{10}(\Omega)^{3/4} \eta(\tau)^{24} \prod_\nu h_\nu(p_2) + \emptyset(t^4) \tag{3.19}$$

Notice that, while each h_ν may not be single-valued on a surface with given spin structure (or without specified spin structures), the product over all ν is single-valued on any surface.

b) The limit of the volume factor $d^6\Omega^{(3)}_{IJ}$

We turn next to the limit of the measure $d^6\Omega^{(3)}_{IJ}$. In the above notation, we have

$$d^6\Omega^{(3)}_{IJ} = d^3\Omega \wedge d\tau \wedge d\tau_1 \wedge d\tau_2 \tag{3.20}$$

We now evaluate $d\tau_1 \wedge d\tau_2$, using the definition of its ingredients,

$$d\tau_1 \wedge d\tau_2 = -\frac{\pi^2}{4}\ t dt \wedge dp_2\ \omega_0(p_1)^2 \Big(\omega_1(p_2)\partial\omega_2(p_2) - \omega_2(p_2)\partial\omega_1(p_2)\Big) \tag{3.21}$$

The combination in parentheses is a holomorphic 3-form in p_2. To evaluate it, we turn to the hyperelliptic representation of Riemann surfaces of genus 2. Let the surface $\Sigma^{(2)}$ be given by

$$s^2 = \prod_{i=1}^{6}(x - u_i) \tag{3.22}$$

Then $z^{J-1}dz/s(z)$, $J = 1, 2$, is a basis of holomorphic differential forms. Let σ_{IJ} be the change of bases matrix from this basis to the basis $\omega^{(2)}_{I_2}(z)$ (which we abbreviate to $\omega_I(z)$ for the rest of the proof of Theorem 1),

$$2\pi i\, \omega_I(z) = \sum_J \sigma_{IJ} \frac{z^{J-1}dz}{s(z)} \tag{3.23}$$

Hence, we have

$$\omega_1(p_2)\partial\omega_2(p_2) - \omega_2(p_2)\partial\omega_1(p_2) = -\frac{1}{4\pi^2}(\det\sigma)\frac{(dp_2)^3}{s(p_2)^2} \tag{3.24}$$

Thus the holomorphic 3-form manifestly has 6 simple zeros precisely at the branch points, exactly as $\prod_\nu h_\nu(z)$. Thus, the p_2-dependence of these two forms is the same.

c) Determining the constant of proportionality

We need to determine the constant of proportionality, which is moduli dependent, and requires several precise coefficients of proportionality between the ϑ-function and the hyperelliptic representation of holomorphic spinors [4]. In the hyperelliptic representation, each of the 6 odd spin structures ν_i corresponds to a branch point u_i, and the one-form $h^2_{\nu_i}(z)$ is proportional to the one-form $(x - u_i)dz/s(z)$. Set

$$h^2_{\nu_i}(z) = \mathcal{N}_{\nu_i}(x - u_i)\frac{dz}{s(z)} \tag{3.25}$$

where $\mathcal{N}_{\nu_i}$ is a moduli dependent constant. Then we have

$$\frac{(dp_2)^3}{s(p_2)^2} = \left(\prod_i \frac{1}{\mathcal{N}_i^{1/2}}\right)\prod_\nu h_\nu(p_2) \tag{3.26}$$

Combining all, we obtain

$$d\tau_1 \wedge d\tau_2 = \frac{1}{16}\; tdt \wedge dp_2\; \omega_0(p_1)^2 \frac{\det\sigma}{\prod_i \mathcal{N}_i^{1/2}} \prod_\nu h_\nu(p_2) \tag{3.27}$$

Next, we have the following two identities

$$
\begin{aligned}
(\det\sigma)^4\vartheta[\delta]^8 &= \prod_{i<j}(a_i-a_j)^2(b_i-b_j)^2 \\
\pi^{24}(\det\sigma)^{12}\vartheta[\delta]^8\Psi_{10}^2 &= \left(\prod_i \mathcal{N}_i^4\right)\prod_{i<j}(a_i-a_j)^2(b_i-b_j)^2
\end{aligned}
\tag{3.28}
$$

Here δ is an even spin structure. In the hyperelliptic representation, it corresponds to a partition of the 6 branch points into two disjoint sets $\{a_1,a_2,a_3\}$ and $\{b_1,b_2,b_3\}$ of three branch points each. The first identity is a classic Thomae formula [16], vol II, §8. To establish the second identity, we make use of the following bilinear ϑ-constants, introduced in [4], equation (2.38)

$$
\mathcal{M}_{\nu_i\nu_j} = \partial_1\vartheta[\nu_i](0,\Omega)\,\partial_2\vartheta[\nu_j](0,\Omega) - \partial_2\vartheta[\nu_i](0,\Omega)\,\partial_1\vartheta[\nu_j](0,\Omega) \tag{3.29}
$$

Solving for $\partial_I\vartheta[\nu_i](0,\Omega)$ from (3.25) and using the formula (3.23) for $\omega_I(z)$, we find

$$
(\det\sigma)\,\mathcal{M}_{\nu_i\nu_j} = 4\pi^2\,\mathcal{N}_{\nu_i}\mathcal{N}_{\nu_j}(u_i-u_j) \tag{3.30}
$$

Taking the products gives

$$
(\det\sigma)^4\mathcal{M}_{12}^2\mathcal{M}_{23}^2\mathcal{M}_{31}^2\mathcal{M}_{45}^2\mathcal{M}_{56}^2\mathcal{M}_{64}^2 = \prod_{i=1}^6\mathcal{N}_{\nu_i}^4\prod_{i<j}(a_i-a_j)^2(b_i-b_j)^2 \tag{3.31}
$$

However, the $\mathcal{M}_{\nu_i\nu_j}^2$ have been determined completely explicitly in terms of ϑ-constants in [4], equation (4.9)

$$
\mathcal{M}_{\nu_i\nu_j} = \pi^4\vartheta[\delta]^2\prod_{k\neq i,j}\vartheta[\nu_i+\nu_j+\nu_k]^2 \tag{3.32}
$$

so that

$$
\mathcal{M}_{12}^2\mathcal{M}_{23}^2\mathcal{M}_{31}^2 = \mathcal{M}_{45}^2\mathcal{M}_{56}^2\mathcal{M}_{64}^2 = \pi^{12}\vartheta[\delta]^4\Psi_{10}(\Omega). \tag{3.33}
$$

Substituting this into (3.31) gives the second identity in (3.28), and (3.28) is now established. Taking the ratio of the two identities in (3.28), we find

$$
\pi^3\,\Psi_{10}(\Omega)^{1/4}\,\det\sigma = \prod_{i=1}^6\mathcal{N}_{\nu_i}^{\frac{1}{2}} \tag{3.34}
$$

Comparing with (3.27), we obtain in this manner the following asymptotics for $d\tau_1\wedge d\tau_2$

$$
d\tau_1\wedge d\tau_2 = \frac{1}{16\pi^3}\,tdt\wedge dp_2\,\omega_0(p_1)^2\frac{1}{\Psi_{10}^{1/4}}\prod_\nu h_\nu(p_2) + \emptyset(t^2). \tag{3.35}
$$

The theorem is now an immediate consequence of (3.19), (3.20), and (3.35). Q.E.D.

3.3 Degeneration limit for the 3-loop bosonic string

We recall that the genus 3 bosonic string measure must satisfy the degeneration constraint (2.16). Since the genus 2 bosonic string measure is given by $c_2\Psi_{10}^{-1}d^3\Omega$, Theorem 1 provides further evidence that the genus 3 bosonic string measure is given by $c_3\Psi_9^{-1}(\Omega^{(3)})d^6\Omega^{(3)}$. In fact, Theorem 1 also dictates what the coefficient of proportionality between the genera 2 and 3 must be

$$c_3 = \frac{c_2}{(2\pi)^6} = \frac{1}{2^6\pi^{18}} \tag{3.36}$$

As another check, we consider the separating degeneration limit of the tachyon amplitude, which is given by the following integral, where $E(z,w)$ is the prime form,

$$\int \left|\frac{dt}{t^2}\right|^2 \prod_{i<j} |E(z_i,z_j)|^{2k_i\cdot k_j} \tag{3.37}$$

The behavior of the prime form when $z_i \in \Sigma^{(2)}$ and $z_j \in \Sigma^{(1)}$ is given by

$$E(z_i,z_j) \to t^{-\frac{1}{2}} E(z_i,p_2)E(p_1,z_j) \tag{3.38}$$

If the sum of the momenta on $\Sigma^{(2)}$ is k,

$$k = \sum_{i,\ z_i\in\Sigma^{(2)}} k_i = -\sum_{j,\ z_j\in\Sigma^{(1)}} k_j \tag{3.39}$$

then we have the following t-dependence,

$$\int \left|\frac{dt}{t^2}\right|^2 |t|^{p^2} \sim \frac{1}{p^2-2} \tag{3.40}$$

which is the expected tachyon pole, with the correct value of the mass squared.

4 The genus 3 superstring measure as a degeneration problem

Using the formula for the genus 2 superstring measure found in [1] and the degeneration formulas of Theorem 1, we can formulate now more concretely the constraints on the genus 3 superstring measure, in the degeneration limit where the worldsheet Σ separates into a torus $\Sigma^{(1)}$ and a surface $\Sigma^{(2)}$ of genus 2. Let Δ be an even genus 3 spin structure. If, in this degeneration, Δ factorizes into two odd spin structures, the leading contribution of order t^{-2} to $d\mu[\Delta](\Omega^{(3)})$ vanishes, and we need not consider this case. Henceforth, we assume that Δ factorizes into two even spin structures, and denote by δ_1 and by $\delta_2 \equiv \delta$ the even spin structures respectively on the torus and on the genus 2 surface $\Sigma^{(2)}$.

Let the genus 3 superstring measure be expressed under the form (1.1), for some density $\Xi_6[\Delta](\Omega^{(3)})$ yet to be determined. Recall that in genus $h = 2$, the superstring measure $d\mu[\delta](\Omega^{(2)})$ was shown to be given by [1, 4]

$$d\mu[\delta](\Omega^{(2)}) = \frac{\vartheta[\delta]^4(0,\Omega^{(2)})\,\Xi_6[\delta](\Omega^{(2)})}{16\pi^6\Psi_{10}(\Omega^{(2)})} \prod_{1\leq I\leq J\leq 2} d\Omega^{(2)}_{IJ} \tag{4.1}$$

The main expression $\Xi_6[\delta](\Omega^{(2)})$ is given in [1], equation (7.1). We shall discuss it further in the next section. The degeneration constraint (2.14), Theorem 1, and the degeneration formulas (3.14) for ϑ-constants imply then that $\Xi_6[\Delta](\Omega^{(3)})$ must satisfy the following limit

$$\lim_{t\to 0} \Xi_6[\Delta](\Omega^{(3)}) = \eta(\Omega^{(1)})^{12}\ \Xi_6[\delta](\Omega^{(2)}). \tag{4.2}$$

This is the condition (*iii*) formulated in the Introduction.

We discuss next the issue of modular invariance for $d\mu[\Delta](\Omega^{(3)})$. The full integrand in the amplitude (2.12) must be invariant under $Sp(6,\mathbf{Z})$. Under the modular transformations (3.1), we have

$$\begin{aligned} \det \mathrm{Im}\,\tilde\Omega^{(3)} &= |\det(C\Omega^{(3)}+D)|^{-2}\,\det \mathrm{Im}\,\Omega^{(3)} \\ \prod_{I\leq J} d\tilde\Omega^{(3)}_{IJ} &= \det(C\Omega^{(3)}+D)^{-4}\prod_{I\leq J} d\Omega^{(3)}_{IJ} \end{aligned} \tag{4.3}$$

At first sight, in genus 3, we have difficulties due to the fact that the expression $\Psi_9(\Omega^{(3)})$ is defined only through its square, $\Psi_{18}(\Omega^{(3)})$. However, the ambiguity in taking square roots here should not be relevant in string theory: for the superstring, Ψ_9 and its conjugate appear in each chiral sector. This is also the case for the heterotic string, since we have seen that Ψ_9 appears in the chiral measure for the bosonic string in the critical dimension, and this is unaffected by compactification. Thus the sign ambiguity in Ψ_9 can be ignored.

In analogy with the genus 2 case, we shall impose then the modular transformation law (*ii*) described in the Introduction on the unknown term $\Xi_6[\Delta](\Omega^{(3)})$ [§]. This condition implies that the superstring chiral measure $d\mu[\Delta](\Omega^{(3)})$ transforms covariantly under modular transformations without any phase factor

$$d\mu[\tilde{\Delta}](\tilde{\Omega}^{(3)}) = \det{(C\Omega^{(3)} + D)^{-5}}\; d\mu[\Delta](\Omega^{(3)}) \tag{4.4}$$

so that a manifestly modular invariant GSO projection is given by

$$\sum_{\Delta} d\mu[\Delta](\Omega^{(3)}). \tag{4.5}$$

This completes our discussion of the three conditions (*i-iii*) formulated in the Introduction for the modular covariant form $\Xi_6[\Delta](\Omega^{(3)})$.

[§]A slightly less restrictive requirement is to allow in (*ii*) an additional phase $\epsilon(M)$ depending only on the modular transformation M, but not on the spin structure Δ. Such additional phases do not affect significantly our subsequent construction of candidates for $\Xi_6[\Delta](\Omega^{(3)})$.

5 Ansätze for the superstring chiral measure

The goal of this section is to construct modular covariant forms in genus 3 satisfying the constraints (*ii*) and (*iii*). The starting point is the expression $\Xi_6[\delta](\Omega^{(2)})$ in genus 2. Our strategy is to find and analyze analogous expressions in genus 3. Although there are several natural analogues, it will turn out that the degeneration condition (*iii*) is quite rigid, and singles out a very small set of candidates.

5.1 The form $\Xi_6[\delta](\Omega^{(2)})$ in genus 2

We begin by recalling the form $\Xi_6[\delta](\Omega^{(2)})$ in genus 2. It was derived directly from the gauge-fixed genus 2 superstring measure, and its original expression was heavily dependent on the fact that the worldsheet had genus 2 (see [1], eq. (7.1)). More recently, two different expressions were found for $\Xi_6[\delta](\Omega^{(2)})$ which do admit generalizations to higher genus [14]. To describe them, recall that a triplet $\{\delta_1,\delta_2,\delta_3\}$ is said to be *asyzygous* if $e(\delta_1,\delta_2,\delta_3)=-1$ (respectively *syzygous* when $+1$), using the usual definitions of the signatures on pairs and triples of spin structures,

$$
\begin{aligned}
\langle\delta_i|\delta_j\rangle \;&\equiv\; \exp(4\pi i(\delta_i'\delta_j''-\delta_i''\delta_j')) \\
e(\delta_1,\delta_2,\delta_3) \;&\equiv\; \langle\delta_1|\delta_2\rangle\,\langle\delta_2|\delta_3\rangle\,\langle\delta_3|\delta_1\rangle
\end{aligned}
\tag{5.1}
$$

More generally, we define as in [14] an N-tuple of spin structures of be *totally asyzygous*, if any triplet of distinct spin structures in the N-tuple is asyzygous

$$
\begin{aligned}
&\{\delta_1,\cdots,\delta_N\}\ \text{totally asyzygous} \\
&\qquad \Leftrightarrow\ \{\delta_i,\delta_j,\delta_k\}\ \text{asyzygous},\quad \text{for all } i,j,k \text{ pairwise distinct.}
\end{aligned}
\tag{5.2}
$$

The notion of totally asyzygous N-tuple is modular invariant, since the cyclic product in (5.1) of relative signatures for a triple of spin structures is.

Returning now to $\Xi_6[\delta](\Omega^{(2)})$, the first alternate expression involves 4-th powers of ϑ-constants (as does the original expression in [1], eq. (7.1)) and is given by

$$
\Xi_6[\delta](\Omega^{(2)}) = -\frac{1}{2}\sum_{\substack{\{\delta,\delta_1,\delta_2,\delta_3\}\\ \text{tot.asyz.}}}\prod_{i=1}^{3}\langle\delta|\delta_i\rangle\ \vartheta[\delta_i](0,\Omega^{(2)})^4
\tag{5.3}
$$

The notation indicates that, for given spin structure δ, the summation runs over all triples $\{\delta_1,\delta_2,\delta_3\}$ such that $\{\delta,\delta_1,\delta_2,\delta_3\}$ forms a totally asyzygous quartet. The second alternate expression for $\Xi_6[\delta](\Omega^{(2)})$ involves only squares of ϑ-constants, but it requires summation over certain sextets of even spin structures. To identify which sextets, we define a sextet

$\{\delta_1, \cdots, \delta_6\}$ of spin structures in genus 2 to be *admissible* if it can be decomposed into three pairs

$$\{\delta_1, \cdots, \delta_6\} = \{\delta_{i_1}, \delta_{i_2}\} \cup \{\delta_{i_3}, \delta_{i_4}\} \cup \{\delta_{i_5}, \delta_{i_6}\}, \tag{5.4}$$

with the union of any two pairs of the decomposition forming a totally asyzygous quartet. For a given spin structure δ, we define the sextet $\{\delta_i\}$ to be δ-*admissible* if it is admissible and it does not contain δ. With this definition, the second alternative expression for $\Xi_6[\delta](\Omega^{(2)})$ is given by

$$\Xi_6[\delta](\Omega^{(2)}) = \frac{1}{2} \sum_{\{\delta_i\}\, \delta-adm.} \epsilon(\delta; \{\delta_i\}) \prod_{i=1}^{6} \vartheta[\delta_i](0, \Omega^{(2)})^2 \tag{5.5}$$

Here the signs $\epsilon(\delta; \{\delta_i\}$ are uniquely related by modular transformations

$$\epsilon(M\delta; \{M\delta_i\}) \prod_{i=1}^{6} \epsilon^2(\delta_i, M) = \epsilon^4(\delta, M)\ \epsilon(\delta; \{\delta_i\}), \qquad M \in Sp(4, \mathbf{Z}), \tag{5.6}$$

where $\epsilon^4(\delta, M)$ is the same factor occurring in the transformation law for $\vartheta^4[\delta]$. An explicit expression for the signs ϵ was given in [14].

5.2 Ansätze in genus 3

The preceding formulas for $\Xi_6[\delta](\Omega)$ in genus 2 suggest several natural extensions to genus 3. We discuss them below. The main issue is whether they can satisfy the desired conditions (*i*), (*ii*), and (*iii*) listed in the Introduction, which are required for any viable Ansatz for the genus 3 superstring chiral measure.

5.2.1 Ansatz in terms of asyzygous quartets of spin structures

The first alternate expression (5.3) for $\Xi_6[\delta](\Omega^{(2)})$ clearly makes sense for arbitrary genus, and in particular for genus 3. Thus we are dealing here with an Ansatz for $\Xi_6[\Delta](\Omega^{(3)})$ involving summations over spin structures $\{\Delta_1, \Delta_2, \Delta_3\}$ which together with the given spin structure Δ, form a totally asyzygous quartet. The modular covariant form which it defines has been studied in [14], where it was denoted by $\Xi_6^\#[\Delta](\Omega^{(3)})$. However, its degeneration limits, as determined in [14], Theorem 5, do not satisfy the degeneration constraint (4.2) for the genus 3 superstring measure. Thus this Ansatz in terms of asyzygous quartets of spin structures must be dropped from contention.

5.2.2 Ansätze in terms of admissible sextets of spin structures

We turn then to several Ansätze which can be viewed as generalizations to genus 3 of the expression (5.5) for $\Xi_6[\delta](\Omega)$ in terms of δ-admissible sextets. First, in complete analogy with the genus case, we define a sextet $\{\Delta_1, \cdots, \Delta_6\}$ to be *admissible* if it can be decomposed as

$$\{\Delta_{i_1}, \Delta_{i_2}\} \cup \{\Delta_{j_1}, \Delta_{j_2}\} \cup \{\Delta_{k_1}, \Delta_{k_2}\} \tag{5.7}$$

with any two pairs constituting a totally asyzygous quartet. Given spin structure Δ, a sextet is said to be Δ-*admissible* if it is admissible, and it does not contain Δ.

Despite the similarity in the definitions, there is in practice a fundamental difference between admissible sextets of spin structures in genus 2 and in genus 3: if

$$s = \{\delta_{i_1}, \delta_{i_2}\} \cup \{\delta_{j_1}, \delta_{j_2}\} \cup \{\delta_{k_1}, \delta_{k_2}\} \tag{5.8}$$

is an admissible sextet in genus 2, then the triplets $\{\delta_{i_\alpha}, \delta_{j_\beta}, \delta_{k_\gamma}\}$, with $\alpha, \beta, \gamma = 1, 2$, are automatically syzygous. Furthermore, if the sextet is δ-admissible, then the following triplet signatures are automatically determined,

$$e(\delta, e_{i_1}, e_{i_2}) = e(\delta, e_{j_1}, e_{j_2}) = e(\delta, e_{k_1}, e_{k_2}) = +1 \tag{5.9}$$

This may easily be inferred by inspection of Table 4 in [14].

This is no longer true for genus 3: in an admissible sextet $\{\Delta_1, \cdots, \Delta_6\}$, the triplets $\{\Delta_{i_\alpha}, \Delta_{j_\beta}, \Delta_{k_\gamma}\}$ need not all be syzygous (or all asyzygous). Thus the admissible sextets in genus 3 fall into 2 categories:

(1) All triplets $\{\Delta_{i_\alpha}, \Delta_{j_\beta}, \Delta_{k_\gamma}\}$ are asyzygous, so that the whole sextet $\{\Delta_1, \cdots, \Delta_6\}$ is *totally asyzygous*;

(2) At least one triplet $\{\Delta_{i_\alpha}, \Delta_{j_\beta}, \Delta_{k_\gamma}\}$ is syzygous. In this case, the relations (5.9) also do not follow from Δ-admissibility. In particular, one has a classification depending on the following signs

$$\begin{aligned}
\rho_1 &= e(\Delta, \Delta_{j_1}, \Delta_{k_1}) \\
\rho_2 &= e(\Delta, \Delta_{k_1}, \Delta_{i_1}) \\
\rho_3 &= e(\Delta, \Delta_{i_1}, \Delta_{j_1})
\end{aligned} \tag{5.10}$$

The four resulting cases, namely $[+++]$, $[++-]$, $[+--]$ and $[---]$ are non-empty and the modular group acts within each case, though not necessarily transitively so. For convenience, we refer to all these cases as cases of *partially asyzygous* sextets.

• *Ansätze in terms of totally asyzygous sextets*

We shall examine two Ansätze, in terms of totally asyzygous sextets with sign assignments.

$$
\begin{aligned}
&\text{(A)} \qquad \Xi_6[\Delta](\Omega^{(3)}) \sim \sum_{\substack{\{\Delta_i\}\text{tot. asyz.}\\ \Delta\notin\{\Delta_i\}}} \epsilon(\Delta;\{\Delta_i\}) \prod_{i=1}^{6} \vartheta[\Delta_i](0,\Omega^{(3)})^2 \\
&\text{(B)} \qquad \Xi_6[\Delta](\Omega^{(3)}) \sim \Bigg(\sum_{\substack{\{\Delta_i\},\{\Delta'_i\}\text{tot. asyz.}\\ \Delta\notin\{\Delta_i\},\{\Delta'_i\}}} \epsilon(\Delta;\{\Delta_i\},\{\Delta'_i\}) \prod_{i=1}^{6} \vartheta[\Delta_i](0,\Omega^{(3)})^2 \vartheta[\Delta'_i](0,\Omega^{(3)})^2 \Bigg)^{1/2}
\end{aligned}
\tag{5.11}
$$

A key issue in these Ansätze is whether sign assignments exist which are consistent with modular transformations. The first Ansatz (A) is simpler, and it will turn out that it does satisfy the degeneration constraint (4.2) if a consistent assignment existed. However, this turns out not to be the case, which is why the second Ansatz (B) is needed. This second Ansatz (B) turns out to be the only viable candidate for $\Xi_6[\Delta](\Omega^{(3)})$ among all the ones examined in the present paper. Its full treatment requires the rest of the paper. We postpone it then to the next section §6, and complete now the discussion of the remaining Ansätze, which involve partially asyzygous sextets.

• *Ansatz in terms of partially asyzygous sextets*

In these remaining cases, the natural Ansätze would be

$$
\Xi_6[\Delta](\Omega^{(3)}) \sim \sum_{[\rho_1,\rho_2,\rho_3]} \epsilon(\Delta;\{\Delta_i\}) \prod_{i=1}^{6} \vartheta[\Delta_i](0,\Omega^{(3)})^2 \tag{5.12}
$$

where the summation would be over all Δ-admissible sextets $\{\Delta_1,\cdots,\Delta_6\}$ with some fixed sign assignment $[\rho_1,\rho_2,\rho_3]$, with not all ρ_i equal to -1.

The first task is to examine whether consistent phase assignments $\epsilon(\Delta;\{\Delta_i\})$ exist. One does this orbit by orbit under the modular group which leaves Δ invariant, and uses the usual modular sign factors in the transformations of ϑ^2. The results are as follows, where we have numbered the genus 3 even spin structures as in Appendix §C of [14],

1. For the cases $[---]$, $[+--]$ and $[++-]$, all orbits produce inconsistent sign assignments and are ruled out;

2. For the case $[+++]$, one orbit with 1680 sextets (generated by sextet $\{2,4,5,6,33,35\}$) and one orbit with 3360 sextets (generated by sextet $\{5,7,12,13,22,30\}$) both generate *consistent sign assignments*;

3. For the case $[+++]$, there is one remaining orbit (generated by sextet $\{2,4,5,9,27,32\}$) which produces an inconsistent sign assignment.

A simple example showing the non-existence of consistent phases for one of these orbits of Δ-admissible, partially asyzygous sextets is given in section §6.2.3 below.

The actual sums in both cases of 2. above are non-vanishing. In the limit where the surface degenerates to a genus 2 times genus 1 surface, both sums converge to the same limit as the form $\Xi_6^{\#}[\delta](\Omega^{(3)})$ of [14], which is inconsistent with the requirement (*iii*) in the Introduction. Thus, even though the sign assignments are consistent, the limits are not and the cases are all ruled out.

Although this analysis rules out a construction of $\Xi_6[\Delta](\Omega^{(3)})$ in terms of partially asyzygous Δ-admissible sextets, it is in principle still possible that an Ansatz for $\Xi_6[\Delta](\Omega^{(3)})^2$ can be obtained in terms of pairs of partially asyzygous Δ-admissible sextets, just as we outlined in the preceding case (B) of totally asyzygous Δ-admissible sextets. However, there does not appear to be any clear way of recapturing $\Xi_6[\delta](\Omega^{(2)})^2$ from the degeneration limits of sums over pairs of partially asyzygous sextets.

5.2.3 Ansatz in terms of dozens of spin structures

Since the previous Ansätze have not produced viable candidates for $\Xi_6[\Delta](\Omega^{(3)})$ itself as a polynomial in ϑ^2, we may ask whether polynomials in ϑ could work,

$$\Xi_6[\Delta](\Omega^{(3)}) = \sum_{\{\Delta_i\}} \epsilon(\Delta; \{\Delta_i\}_{1\leq i\leq 12}) \prod_{i=1}^{12} \vartheta[\Delta_i](0,\Omega^{(3)}), \tag{5.13}$$

where the summation runs over a suitable set of dozens $\{\Delta_1, \cdots, \Delta_{12}\}$ of spin structures. Here the expression for $\Xi_6[\delta](\Omega)$ in genus 2 provides little guidance for choosing this set. There are conceivably many possibilities. But, given the good degeneration limits of totally asyzygous sextets, it is natural to consider products of pairs of totally asyzygous sextets,

$$\Xi_6'[\Delta](\Omega^{(3)}) = \sum_{\{s_1,s_2\}\in\mathcal{Q}_{pq}} \epsilon(\Delta, s_1, s_2) \prod_{i=1}^{12} \vartheta[\Delta_i](0,\Omega^{(3)}) \tag{5.14}$$

where the sum is over pairs of totally asyzygous sextets,

$$\begin{aligned} s_1 &= \{\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6\} \\ s_2 &= \{\Delta_7, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}\} \end{aligned} \tag{5.15}$$

and $\mathcal{Q}_{pq}$ denote the different orbits of Δ-admissible pairs of totally asyzygous sextets under the modular subgroup leaving Δ invariant. The orbits $\mathcal{Q}_{pq}$ are described in detail in section §6.3.2 below.

Clearly, this construction can make sense only for orbits $\mathcal{Q}_{pq}$ for which the phases $\epsilon(\Delta, s_1, s_2)^2$ can be consistently defined. But this problem was already solved (by computer) in the treatment of the Ansatz (B) in terms of pairs of totally asyzygous sextets (see section §6.3.2 and subsequent discussions). It was found that consistent phases exist for the orbits $\mathcal{Q}_{01}$, $\mathcal{Q}_{02}$, $\mathcal{Q}_{13}$, $\mathcal{Q}_{20}$ $\mathcal{Q}_{21}$, $\mathcal{Q}_{22}$, and $\mathcal{Q}_{23}$ but not for the orbits $\mathcal{Q}_{11}$, $\mathcal{Q}_{12}$, and $\mathcal{Q}_{3}$. A computer calculation shows, however, that in none of these orbits the sign $\epsilon(\Delta, s_1, s_2)$ can actually be consistently defined. Thus this particular Ansatz for $\Xi_6[\Delta](\Omega^{(3)})$ is also ruled out.

6 The Ansätze in terms of totally asyzygous sextets

To determine the degeneration behavior of the genus 3 candidates (A) and (B), we need to determine the degeneration behavior of genus 3 totally asyzygous sextets of even spin structures.

6.1 Degenerations of totally asyzygous sextets

The basic fact is the following:

Lemma 1. *Let $\{\Delta_i\}_{1\leq i\leq 6}$ be a sextet of genus 3 even spin structures. Assume that it is totally asyzygous and that each Δ_i degenerates into even spin structures in genera 2 and 1. Let $\delta_1, \cdots, \delta_6$ be the 6 genus 2 even spin structures which arise in this manner. Then the sextet $\{\delta_1, \cdots, \delta_6\}$ is an admissible sextet of genus 2 even spin structures in the sense defined above, that is, it can be divided into three pairs, the union of any two defines a totally asyzygous quartet. Furthermore, we have*

$$\prod_{i=1}^{6} \vartheta[\Delta_i](0, \Omega^{(3)})^2 \to 2^4 \eta(\Omega^{(1)})^{12} \prod_{i=1}^{6} \vartheta[\delta_i](0, \Omega^{(2)})^2 \tag{6.1}$$

Proof. We recall that there exist no totally asyzygous quintets at genus 2 (and thus no genus 2 totally asyzygous sextuplets etc). This can be seen by direct inspection of the tables of asyzygies in genus 2 provided in [14]. Let $\mu_1, \cdots, \mu_6$ be the genus 1 spin structures arising from the degeneration of $\Delta_1, \cdots, \Delta_6$. By assumption, they are even. We examine in turn all possible arrangements for $\mu_1, \cdots, \mu_6$:

- Assume that $\mu_1, \cdots, \mu_6$ take at most 2 distinct values amongst the 3 possible even spin structures at genus 1. Then, it follows that $e(\mu_i, \mu_j, \mu_k) = +1$ for any triplet of μ's arising in the sextet. For the genus 3 sextet to be totally asyzygous, the genus 2 sextuplet $\delta_1, \cdots, \delta_6$ must be totally asyzygous, but this is impossible.

- Assume that five of the six $\mu_1, \cdots, \mu_6$ (say $\mu_1, \cdots, \mu_5$ for definiteness) take at most 2 distinct values amongst the 3 possible even spin structures at genus 1. Then $e(\mu_i, \mu_j, \mu_k) = +1$ for $1 \leq i, j, k \leq 5$. For the genus 3 sextet to be totally asyzygous, the genus 2 quintet $\delta_1, \cdots, \delta_5$ must be totally asyzygous, but this is impossible.

- The only remaining possibility is that amongst the six $\mu_1, \cdots, \mu_6$, each of the 3 distinct genus 1 even spin structures (which we denote μ_2, μ_3, μ_4 by slight abuse of notation) occurs precisely twice.

Thus, up to permutations of the μ's, we have

$$\Delta_1 = \begin{pmatrix} \delta_1 \\ \mu_2 \end{pmatrix} \qquad \Delta_3 = \begin{pmatrix} \delta_3 \\ \mu_3 \end{pmatrix} \qquad \Delta_5 = \begin{pmatrix} \delta_5 \\ \mu_4 \end{pmatrix}$$
$$\Delta_2 = \begin{pmatrix} \delta_2 \\ \mu_2 \end{pmatrix} \qquad \Delta_4 = \begin{pmatrix} \delta_4 \\ \mu_3 \end{pmatrix} \qquad \Delta_6 = \begin{pmatrix} \delta_6 \\ \mu_4 \end{pmatrix} \tag{6.2}$$

It is clear that the quartets $\{\delta_1\ \delta_2\ \delta_3\ \delta_4\}$, $\{\delta_1\ \delta_2\ \delta_5\ \delta_6\}$, $\{\delta_3\ \delta_4\ \delta_5\ \delta_6\}$, are totally asyzygous, and that they are the only totally asyzygous quartets within $\{\delta_1, \cdots, \delta_6\}$. This proves the first part of Lemma 1. The second part follows immediately from the degeneration formulas for ϑ-constants and from the identity (3.17). Q.E.D.

6.2 Orbits of sextets

Since the genus 2 expression $\Xi_6[\delta](\Omega^{(2)})$ is built from genus 2 admissible sextets, Lemma 1 shows that asyzygous sextets have the potential to produce a form $\Xi_6[\Delta](\Omega^{(3)})$ tending to $\Xi_6[\delta](\Omega^{(2)})$ in the degeneration limit. Fix an even spin structure Δ. In analogy with the genus 2 case, we define a Δ-admissible sextet of even spin structures to be a totally asyzygous sextet $\{\Delta_i\}$ not containing Δ. We can restrict then the sextets entering the candidate for $\Xi_6[\Delta](\Omega^{(3)})$ to the Δ-admissible ones. This justifies the form given in (5.11) for the Ansätze (A) and (B).

We need to consider the degenerations of Δ-admissible sextets $\{\Delta_i\}$. We can assume that

$$\Delta = \begin{pmatrix} \delta \\ \mu \end{pmatrix} \tag{6.3}$$

with both lower genus spin structures μ_i and δ even, since otherwise Δ will not contribute to the leading asymptotics. Let $\{\delta_i\}$ be the sextet of genus 2 spin structures obtained by factoring $\{\Delta_i\}$. We can assume that they are all even, since otherwise $\{\Delta_i\}$ will again not contribute to the leading asymptotics. Now Lemma 1 guarantees that the sextet $\{\delta_i\}$ is admissible in the genus 2 sense. However, the condition $\Delta \notin \{\Delta_i\}$ does not guarantee that $\delta \notin \{\delta_i\}$, i.e., the Δ-admissibility of the genus 3 sextet $\{\Delta_i\}$ does not guarantee the δ-admissibility of the genus 2 sextet $\{\delta_i\}$. Thus we have to analyze the contributions of genus 2 admissible sextets which are not δ-admissible. We also have to determine the exact multiplicities with which δ-admissible and δ-not admissible sextets occur in the degeneration of an $Sp(6, \mathbf{Z})$ orbit of Δ-admissible sextets in genus 3. The first issue is addressed by Lemma 2 below. The second issue will be addressed by a computer listing of all possibilities. The results will be described in subsequent sections.

6.2.1 Orbits of admissible sextets in genus 2

The list of admissible sextets in genus 2 is provided in [14], Table 4. By a simple inspection of that list and the actions of modular transformations in the same table, we find that

• There are no totally asyzygous quintets, and a fortiori, no totally asyzygous sextets in genus 2;

• In genus 2, there are 15 admissible sextets. The group $Sp(4, \mathbf{Z})$ acts transitively on the set of admissible sextets;

• Given a genus 2 even spin structure δ, there are always exactly 6 sextets which are admissible, and 9 which are not. We denote these sets of sextets by $s[\delta]$ and $s^c[\delta]$ respectively

$$
\begin{aligned}
s[\delta] &= \Big\{\{\delta_i\} \text{ admissible sextet } ; \ \delta \notin \{\delta_i\}\Big\} \\
s^c[\delta] &= \Big\{\{\delta_i\} \text{ admissible sextet } ; \ \delta \in \{\delta_i\}\Big\};
\end{aligned}
\tag{6.4}
$$

• Let $Sp[\delta](4, \mathbf{Z})$ be the subgroup of $Sp(4, \mathbf{Z})$ fixing δ. Then $Sp[\delta](4, \mathbf{Z})$ acts transitively on both $s[\delta]$ and $s^c[\delta]$. In particular, if the phases $\epsilon(\delta; \{\delta_i\})$ satisfy the transformation (**??**), then all the phases in each orbit $s[\delta]$ or $s^c[\delta]$ are uniquely determined by the phase of any single element inside $s[\delta]$ and $s^c[\delta]$.

Lemma 2. *Let δ be a fixed genus 2 even spin structure. Assume that the phases $\epsilon(\delta; \{\delta_i\})$ satisfy the condition (5.6) for all $M \in Sp[\delta](4, \mathbf{Z})$. Then we have*

$$
\sum_{\{\delta_i\} \in s[\delta]} \epsilon(\delta; \{\delta_i\}) \prod_{i=1}^{6} \vartheta[\delta_i]^2 = \pm\, 2\, \Xi_6[\delta](\Omega^{(2)}) \tag{6.5}
$$

$$
\sum_{\{\delta_i\} \in s^c[\delta]} \epsilon(\delta; \{\delta_i\}) \prod_{i=1}^{6} \vartheta[\delta_i]^2 = 0. \tag{6.6}
$$

The $\pm$ sign in the first identity is a consequence of the fact that the phases $\epsilon(\delta; \{\delta_i\})$ in each orbit $s[\delta]$ or $s^c[\delta]$ are determined only up to a global sign.

Proof. The first identity in (6.5) is just a reformulation of (5.5), and was proved in [14]. To establish the second identity, we go to the hyperelliptic representation.

Let $s^2 = \prod_{i=1}^{6}(x - p_i)$ be a hyperelliptic representation for the surface $\Sigma^{(2)}$ ¶. As before, we identify the spin structure δ with a partition of the 6 branch points into two sets of 3

¶The branch points p_i here should not be confused with the punctures p_1 and p_2 in the degeneration construction of §2. The notation p_i for the branch points is in accord with [14], which is used heavily in the proof of Lemma 2.

branch points each, say $\delta \sim \{a_1, a_2, a_3\} \cup \{b_1, b_2, b_3\}$. The Thomae formula (for genus 2) takes the following form,

$$\vartheta[\delta]^2 = \epsilon C x_{a_1 a_2} x_{a_2 a_3} x_{a_3 a_1} x_{b_1 b_2} x_{b_2 b_3} x_{b_3 b_1} \qquad x_{p_i p_j} = \sqrt{p_i - p_j} \tag{6.7}$$

Here, $\epsilon^4 = 1$, and C is δ-independent. Actually, we need the explicit correspondence only for the sextets themselves. Given the normalization of a single sextet, the correspondences for all others may be derived using the action of modular transformations on both sides. We fix the expression for one sextet, say (125690), to be C^6, and determine the hyperelliptic expressions for the others by modular transformations,

$$\begin{aligned}
t_1 \equiv +(125690) &= +(p_1 - p_6)(p_2 - p_4)(p_3 - p_5)\ C^6 V(p_i) \\
t_2 \equiv +(137890) &= -(p_1 - p_3)(p_2 - p_5)(p_4 - p_6)\ C^6 V(p_i) \\
t_3 \equiv +(145678) &= -(p_1 - p_4)(p_2 - p_3)(p_5 - p_6)\ C^6 V(p_i) \\
t_4 \equiv +(124580) &= -(p_1 - p_4)(p_2 - p_6)(p_3 - p_5)\ C^6 V(p_i) \\
t_5 \equiv +(134670) &= +(p_1 - p_5)(p_2 - p_3)(p_4 - p_6)\ C^6 V(p_i) \\
t_6 \equiv +(123689) &= -(p_1 - p_6)(p_2 - p_5)(p_3 - p_4)\ C^6 V(p_i) \\
t_7 \equiv -(134589) &= +(p_1 - p_4)(p_2 - p_5)(p_3 - p_6)\ C^6 V(p_i) \\
t_8 \equiv -(124679) &= -(p_1 - p_6)(p_2 - p_3)(p_4 - p_5)\ C^6 V(p_i) \\
t_9 \equiv -(123570) &= +(p_1 - p_2)(p_4 - p_6)(p_3 - p_5)\ C^6 V(p_i) \\
t_{10} \equiv +(235678) &= +(p_1 - p_2)(p_3 - p_4)(p_5 - p_6)\ C^6 V(p_i) \\
t_{11} \equiv +(247890) &= -(p_1 - p_3)(p_2 - p_6)(p_4 - p_5)\ C^6 V(p_i) \\
t_{12} \equiv -(234579) &= +(p_1 - p_2)(p_3 - p_6)(p_4 - p_5)\ C^6 V(p_i) \\
t_{13} \equiv +(234680) &= -(p_1 - p_5)(p_2 - p_6)(p_3 - p_4)\ C^6 V(p_i) \\
t_{14} \equiv +(345690) &= +(p_1 - p_5)(p_2 - p_4)(p_3 - p_6)\ C^6 V(p_i) \\
t_{15} \equiv +(567890) &= -(p_1 - p_3)(p_2 - p_4)(p_5 - p_6)\ C^6 V(p_i)
\end{aligned} \tag{6.8}$$

The omnipresent factor V is the Vandermonde polynomial

$$V(p_i) \equiv \prod_{1 \le i < j \le 6} x_{ij}^2 = \prod_{1 \le i < j \le 6} (p_i - p_j) \tag{6.9}$$

Under a permutation of the branch points, $V(p_i)$ is multiplied by the signature of this permutation. The following modular transformations were used to establish these signs,

$$\begin{array}{lll}
\Sigma(t_1) = +t_2 & M_3(t_4) = +t_8 & S(t_{14}) = +t_{11} \\
T(t_1) = +t_3 & S(t_8) = +t_7 & M_1(t_8) = -t_{12} \\
S(t_3) = +t_6 & T(t_7) = +t_9 & M_3(t_{12}) = +t_{13} \\
T(t_6) = +t_5 & M_1(t_3) = -t_{10} & S(t_{13}) = +t_{15} \\
\Sigma(t_5) = +t_4 & T(t_{10}) = +t_{14} &
\end{array} \tag{6.10}$$

Taking into account the behavior of V under permutations, modular invariance determines the relative signs in the sums over $s[\delta]$ and $s^c[\delta]$. Working this out for one of the spin structures, say $\delta = \delta_1$ gives the following explicit formulas,

$$
\begin{aligned}
\sum_{\{\delta_i\}\in s[\delta_1]} \epsilon(\delta_1; \{\delta_i\}) \prod_{i=1}^{6} \vartheta[\delta_i]^2 &= 2t_{12} + 2t_{13} + 2t_{15} = -2t_{10} - 2t_{11} - 2t_{14} \\
&= 2\Xi_6[\delta_1] \\
\sum_{\{\delta_i\}\in s^c[\delta_1]} \epsilon(\delta_1; \{\delta_i\}) \prod_{i=1}^{6} \vartheta[\delta_i]^2 &= \sum_{i=1}^{9} t_i = 0
\end{aligned}
\tag{6.11}
$$

The modular covariance properties then yield these results for all spin structures δ and thus completes the proof of the theorem. Q.E.D.

6.2.2 Orbits of admissible sextets in genus 3 (totally asyzygous sextets)

We list here a number of results on the modular transformations of asyzygous multiplets $\{\Delta_i\}$ in genus 3, all of which have been proven by computer calculations.

- The sets of all asyzygous quartets, quintets and sextets transform transitively under the full modular group acting on characteristics;

- There are 5040 totally asyzygous quartets, 2016 totally asyzygous quintets, 336 totally asyzygous sextets, and no totally asyzygous septets;

- The set of all asyzygous sextets that do not contain a given spin structure Δ transforms transitively under the modular subgroup $Sp[\Delta](6, \mathbf{Z})$ leaving Δ invariant. In analogy with the genus 2 case, we denote by $S[\Delta]$ the set of asyzygous sextets not containing a spin structure Δ. For any Δ, $S[\Delta]$ consists of 280 elements;

- Upon factorization, each sextet $\{\Delta_i\}$ of genus 3 spin structures produces a sextet $\{\delta_i\}$ of genus 2 spin structures. Consider the set of the 336 sextets $\{\delta_i\}$ of genus 2 spin structures which are obtained from factorization from the set of all 336 asyzygous sextets in genus 3. Then the set of such $\{\delta_i\}$ can be divided into 246 sextets which contain at least some odd spin structure, together with 6 copies of all 15 genus 2 admissible sextets;

- Similarly, let Δ factorize into a genus 1 and a genus 2 spin structure δ as in (6.3), and consider the set of all genus 2 sextets $\{\delta_i\}$ arising from factorization of the 280 Δ-admissible genus 3 sextets in $S[\Delta]$. Then the set of such $\{\delta_i\}$ can be divided into 208 sextets which contain at least some odd spin structures, together with 6 copies of $s[\delta]$ and 4 copies of $s^c[\delta]$.

We can now consider the first Ansatz in (5.11) for $\Xi_6[\Delta](\Omega^{(3)})$, where the summation is over the set $S[\Delta]$ of Δ-admissible sextets. For $\Xi_6[\Delta](\Omega^{(3)})$ to transform as in (*ii*), we

impose the analogous condition to (5.6) in genus 3

$$\epsilon(M\Delta;\{M\Delta_i\})\ \prod_{i=1}^{6}\epsilon^2(\Delta_i,M)=\epsilon^4(\Delta,M)\ \epsilon(\delta;\{\Delta_i\}),\qquad M\in Sp(6,\mathbf{Z}), \tag{6.12}$$

Restricted to $M \in Sp[\Delta](6,\mathbf{Z})$, this implies that all the phases $\epsilon(M\Delta;\{M\Delta_i\})$ in the first Ansätz uniquely determine one another. Assuming the existence of such a consistent assignment of phases, the expression in the first Ansätz is then uniquely determined up to a global $\pm$ sign. The $Sp[\Delta](6,\mathbf{Z})$ consistency of phases implies the $Sp[\delta](4,\mathbf{Z})$ consistency of phases. Thus Lemmas 1 and 2 apply. Together with the numerology for the degeneration of the orbit $S[\Delta]$ found above, we obtain

$$\lim_{t\to 0}\sum_{\{\Delta_i\}\in S[\Delta]}\epsilon(\Delta;\{\Delta_i\})\prod_{i=1}^{6}\vartheta[\Delta_i]^2(0,\Omega^{(3)})=6\cdot 2^4\,\eta(\Omega^{(1)})^{12}\,\Xi_6[\delta](\Omega^{(2)}) \tag{6.13}$$

Here, we assume that all 6 copies of $s[\delta]$ obtained in factoring $S[\Delta]$ lead to contributions of the same sign. However, there is a more severe obstruction to the Ansatz (A):

• There does not exist a phase assignment $\epsilon(\Delta;\{\Delta_i\})$ satisfying the condition (6.12) and the sextets are totally asyzygous. This is in marked contrast with the genus 2 case, where the phases $\epsilon(\delta;\{\delta_i\})$ satisfying (5.6) do exist. A counterexample in genus 3 is obtained by considering the following Δ_1-admissible sextet,‖

$$s_1=(\Delta_2,\Delta_8,\Delta_{14},\Delta_{16},\Delta_{25},\Delta_{30}) \tag{6.14}$$

and the action of the composite modular transformation A_1B_4. From Table 6 of [14], it is clear that A_1B_4 leaves $\Delta_1,\Delta_3,\Delta_4,\Delta_6$ invariant and maps $\Delta_2\leftrightarrow\Delta_5$. Thus, the Δ-admissible sextet s_1, as a whole, is invariant under A_1B_4. The sign factor is also easily computed, using

$$\epsilon^2(\Delta_i,A_1B_4)=\epsilon^2(B_4\Delta_i,A_1)\times\epsilon^2(\Delta_i,B_4)=e^{4\pi i(\Delta_i)'_1(\Delta_i)'_2} \tag{6.15}$$

and we find

$$\epsilon^2(\Delta_i,A_1B_4)=+1\qquad i=2,8,14,16,25\qquad \epsilon^2(\Delta_{30},A_1B_4)=-1 \tag{6.16}$$

But then the sextet contribution changes sign under a transformation that leaves the sextet invariant, which means to no consistent sign can be defined.

‖Throughout, we shall use the nomenclature for genus 3 spin structures and modular transformations given in Appendix C of [14].

6.2.3 Orbits of admissible sextets in genus 3 (partially asyzygous sextets)

A consistent phase assignment is also lacking in this case. A counterexample in genus 3 is obtained by considering the following Δ_1-admissible sextet,

$$s_2 = (\Delta_2, \Delta_6, \Delta_8, \Delta_{18}, \Delta_{29}, \Delta_{36}) \tag{6.17}$$

The modular transformation $A_6B_6A_6B_6$ leaves each of the spin structures in s_2, and thus the entire sextet, invariant. The signs accompanying the transformation are easily computed, using

$$\begin{aligned} \epsilon^2(\Delta_i, A_6B_6A_6B_6) &= +1 \qquad\qquad i = 2, 8, 18 \\ \epsilon^2(\Delta_i, A_6B_6A_6B_6) &= -1 \qquad\qquad i = 6, 29, 36 \end{aligned} \tag{6.18}$$

But then the sextet contribution changes sign under a transformation that leaves the sextet invariant, which means to no consistent sign can be defined.

6.3 Orbits of pairs of sextets

In the preceding section, we have seen sums over Δ-admissible sextets are not consistent with the modular transformation (6.12). Thus we cannot construct $\Xi_6[\Delta](\Omega^{(3)})$ directly by the Ansatz (A). In this section, we shall show that certain sums over *pairs of sextets* do admit consistent phase assignments, and that carefully chosen sums do lead to viable candidates for $\Xi_6[\Delta](\Omega^{(3)})^2$.

6.3.1 Orbits of pairs of admissible sextets in genus 2

Fix an external genus 2 even spin structure δ. Our first task is to identify the orbits of pairs δ-admissible sextets under $Sp[\delta](4, \mathbf{Z})$. Clearly, for each integer p, the subset of pairs $\{\delta_i\}$, $\{\delta_i'\}$ with p common spin structures is invariant under $Sp[\delta](4, \mathbf{Z})$. For δ-admissible pairs of sextets, there is a finer partition which does give precisely all the orbits under $Sp[\delta](4, \mathbf{Z})$:

$$\begin{aligned} Q_p^{0,0}[\delta] &= \{(\{\delta_i\}, \{\delta_i'\}) \in s[\delta] \times s[\delta];\ \#(\{\delta_i\} \cap \{\delta_i'\}) = p\} \\ Q_p^{0,1}[\delta] &= \{(\{\delta_i\}, \{\delta_i'\}) \in s[\delta] \times s^c[\delta];\ \#(\{\delta_i\} \cap \{\delta_i'\}) = p\} \\ Q_p^{1,0}[\delta] &= \{(\{\delta_i\}, \{\delta_i'\}) \in s^c[\delta] \times s[\delta];\ \#(\{\delta_i\} \cap \{\delta_i'\}) = p\} \\ Q_p^{1,1}[\delta] &= \{(\{\delta_i\}, \{\delta_i'\}) \in s^c[\delta] \times s^c[\delta];\ \#(\{\delta_i\} \cap \{\delta_i'\}) = p\} \end{aligned} \tag{6.19}$$

By inspecting the table of admissible sextets in genus 2, we find that only the values $p = 3, 4$ and 6 produce non-empty sets $Q_p^{a,b}[\delta]$. The sizes of the orbits $Q_p^{a,b}[\delta]$ are given

by $\# Q_3^{0,0}[\delta] = 12$, $\# Q_4^{0,0}[\delta] = 18$, $\# Q_6^{0,0}[\delta] = 6$, $\# Q_3^{0,1}[\delta] = \# Q_3^{1,0}[\delta] = 36$, $\# Q_4^{0,1}[\delta] = \# Q_4^{1,0}[\delta] = 18$, $\# Q_3^{1,1}[\delta] = 36$, $\# Q_4^{1,1}[\delta] = 36$, $\# Q_6^{1,1}[\delta] = 9$, which does add up to $15^2 = 225$. In this counting, the pairs of sextets have been viewed as ordered pairs. For later purposes, it is preferrable to count unordered pairs, in which case the sizes of the orbits $Q_p^{a,b}[\delta]$ become

$$\begin{array}{ccc} Q_3^{0,0} = 6 & Q_3^{0,1} = 36 & Q_3^{1,1} = 18 \\ Q_4^{0,0} = 9 & Q_4^{0,1} = 18 & Q_4^{1,1} = 18 \\ Q_6^{0,0} = 6 & Q_6^{0,1} = 0 & Q_6^{1,1} = 9 \end{array} \tag{6.20}$$

To each orbit $Q_p^{a,b}[\delta]$, we can associate the following polynomial in ϑ-constants

$$F_p^{a,b}[\delta] = \sum_{(\{\delta_i\},\{\delta'_i\}) \in Q_p^{a,b}[\delta]} \epsilon_p^{a,b}(\delta;\{\delta_i\},\{\delta'_i\}) \prod_{i=1}^{6} \vartheta[\delta_i]^2 \prod_{i=1}^{6} \vartheta[\delta'_i]^2 \tag{6.21}$$

where the phases $\epsilon(\delta;\{\delta_i\},\{\delta'_i\})$ are required to satisfy

$$\epsilon(M\delta;\{M\delta_i\},\{M\delta'_i\}) \prod_{i=1}^{6} \epsilon^2(\delta_i, M) \prod_{i=1}^{6} \epsilon^2(\delta'_i, M) = \epsilon(\delta;\{\delta_i\},\{\delta'_i\}) \tag{6.22}$$

Since $Q_p^{a,b}[\delta]$ are orbits of $Sp[\delta](4,\mathbf{Z})$, the phases $\epsilon(\delta;\{\delta_i\},\{\delta'_i\})$ completely determine each other within $Q_p^{a,b}[\delta]$. We also find, by computer inspection, that a consistent assignment of phases $\epsilon(\delta;\{\delta_i\},\{\delta'_i\})$ exist for each $Q_p^{a,b}[\delta]$. Thus the expressions $F_p^{a,b}[\delta]$ exist, and are uniquely determined by a single normalizing sign. We shall define this normalizing sign below.

Remarkably, the expressions $F_p^{a,b}[\delta]$ can be expressed very simply in terms of $\Xi_6[\delta](\Omega^{(2)})$ and two other polynomials in ϑ-constants, defined by

$$\begin{array}{ll} F[\delta_1] \equiv \sum_{i \in s[\delta_1]} t_i^2 & s[\delta_1] = \{10,11,12,13,14,15\} \\ F^c[\delta_1] \equiv \sum_{i \in s^c[\delta_1]} t_i^2 & s^c[\delta_1] = \{1,2,3,4,5,6,7,8,9\} \end{array} \tag{6.23}$$

Then we have

Lemma 3. *Let the normalizing signs for $F_p^{a,b}[\delta]$ be defined by the equation (6.28). Then the expressions $F_p^{a,b}[\delta]$ are given by*

$$\begin{array}{lll} F_3^{0,0}[\delta_1] = \Xi_6[\delta_1]^2 - F[\delta_1]/2 & F_3^{0,1}[\delta_1] = -F^c[\delta_1] & F_3^{1,1}[\delta_1] = F^c[\delta_1]/2 \\ F_4^{0,0}[\delta_1] = -\Xi_6[\delta_1]^2 & F_4^{0,1}[\delta_1] = F^c[\delta_1] & F_4^{1,1}[\delta_1] = -F^c[\delta_1] \\ F_6^{0,0}[\delta_1] = F[\delta_1] & & F_6^{1,1}[\delta_1] = +F^c[\delta_1] \end{array} \tag{6.24}$$

Proof. The following relations were established earlier,

$$\Xi_6[\delta_1] = t_{10} + t_{11} + t_{14} = -t_{12} - t_{13} - t_{15} \tag{6.25}$$

$$0 = t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8 + t_9 \tag{6.26}$$

The relation (6.25) was established as a step in the proof of the alternative form (5.5) of $\Xi_6[\delta](\Omega^{(2)})$ in [14]. The relation (6.26) is a reformulation of the second identity in Lemma 2. Additional "rearrangement" formulas are as follows,

$$\begin{array}{rcccl}
t_1 &=& t_2 + t_3 + t_5 + t_7 &=& t_{14} + t_{15} \\
t_2 &=& t_1 + t_3 + t_4 + t_8 &=& t_{11} + t_{15} \\
t_3 &=& t_1 + t_2 + t_6 + t_9 &=& t_{10} + t_{15} \\
t_4 &=& t_2 + t_5 + t_6 + t_8 &=& t_{11} + t_{13} \\
t_5 &=& t_1 + t_4 + t_6 + t_7 &=& t_{13} + t_{14} \\
t_6 &=& t_3 + t_4 + t_5 + t_9 &=& t_{10} + t_{13} \\
t_7 &=& t_1 + t_5 + t_8 + t_9 &=& t_{12} + t_{14} \\
t_8 &=& t_2 + t_4 + t_7 + t_9 &=& t_{12} + t_{11} \\
t_9 &=& t_3 + t_6 + t_7 + t_8 &=& t_{12} + t_{10}
\end{array} \tag{6.27}$$

and They follow directly from the hyperelliptic representation; the equivalences are under the relations (6.25, 6.26).

We define now the normalizing signs for $F_p^{a,b}[\delta]$ promised earlier. Writing $e_p^{a,b}(\delta_1; t_i, t_j) = e_p^{a,b}[\delta](i,j)$ for simplicity, they are given by

$$\begin{array}{lll}
\epsilon_3^{0,0}[\delta_1](10,11) = +1 & \epsilon_3^{0,1}[\delta_1](1,10) = +1 & \epsilon_3^{1,1}[\delta_1](1,2) = +1 \\
\epsilon_4^{0,0}[\delta_1](10,13) = +1 & \epsilon_4^{0,1}[\delta_1](1,14) = +1 & \epsilon_4^{1,1}[\delta_1](1,4) = +1 \\
\epsilon_6^{0,0}[\delta_1](10,10) = +1 & & \epsilon_6^{1,1}[\delta_1](1,1) = +1
\end{array} \tag{6.28}$$

The resulting polynomials are then as follows,

$$\begin{array}{rcl}
F_3^{0,0}[\delta_1](t) &=& +t_{10}t_{11} + t_{10}t_{14} + t_{11}t_{14} + t_{12}t_{13} + t_{12}t_{15} + t_{13}t_{15} \\
F_4^{0,0}[\delta_1](t) &=& +(t_{10} + t_{11} + t_{14})(t_{12} + t_{13} + t_{15}) \\
F_6^{0,0}[\delta_1](t) &=& +t_{10}^2 + t_{11}^2 + t_{12}^2 + t_{13}^2 + t_{14}^2 + t_{15}^2 \\
F_3^{0,1}[\delta_1](t) &=& +t_1(t_{10} + t_{11} + t_{12} + t_{13}) + t_2(t_{10} + t_{12} + t_{13} + t_{14})
\end{array}$$

$$
\begin{aligned}
& +t_3(t_{11}+t_{12}+t_{13}+t_{14})+t_4(t_{10}+t_{12}+t_{14}+t_{15}) \\
& +t_5(t_{10}+t_{11}+t_{12}+t_{15})+t_6(t_{11}+t_{12}+t_{14}+t_{15}) \\
& +t_7(t_{10}+t_{11}+t_{13}+t_{15})+t_8(t_{10}+t_{13}+t_{14}+t_{15}) \\
& +t_9(t_{11}+t_{13}+t_{14}+t_{15}) \\
F_4^{0,1}[\delta_1](t) &= +t_1(t_{14}+t_{15})+t_2(t_{11}+t_{15})+t_3(t_{10}+t_{15}) \\
& +t_4(t_{11}+t_{13})+t_5(t_{13}+t_{14})+t_6(t_{10}+t_{13}) \\
& +t_7(t_{12}+t_{14})+t_8(t_{11}+t_{12})+t_9(t_{10}+t_{12}) \\
F_3^{1,1}[\delta_1](t) &= +t_1t_2+t_1t_3+t_1t_5+t_1t_7+t_2t_3+t_2t_4+t_2t_8+t_3t_6+t_3t_9 \\
& +t_4t_5+t_4t_6+t_4t_8+t_5t_6+t_5t_7+t_6t_9+t_7t_8+t_7t_9+t_8t_9 \\
F_4^{1,1}[\delta_1](t) &= +t_1t_4+t_1t_6+t_1t_8+t_1t_9+t_2t_5+t_2t_6+t_2t_7+t_2t_9+t_3t_4 \\
& +t_3t_5+t_3t_7+t_3t_8+t_4t_7+t_4t_9+t_5t_8+t_5t_9+t_6t_7+t_6t_8 \\
F_6^{1,1}[\delta_1](t) &= +t_1^2+t_2^2+t_3^2+t_4^2+t_5^2+t_6^2+t_7^2+t_8^2+t_9^2 \qquad (6.29)
\end{aligned}
$$

The above expressions for F can be recast in the following, more systematic way,

$$
\begin{aligned}
F_p^{0,0}[\delta_1](t) &= \sum_{\substack{\#(i\cap j)=p, \\ i\leq j; i,j\in s[\delta_1]}} t_i t_j && p=3,4,6 \\
F_p^{0,1}[\delta_1](t) &= \sum_{\substack{\#(i\cap j)=p, \\ i\in s[\delta_1], j\in s^c[\delta_1]}} t_i t_j && p=3,4 \\
F_p^{1,1}[\delta_1](t) &= \sum_{\substack{\#(i\cap j)=p, \\ i\leq j, i,j\in s^c[\delta_1]}} t_i t_j && p=3,4,6 \qquad (6.30)
\end{aligned}
$$

Using the relations (6.25), (6.26), and (6.27), we can reduce these expressions to the linear combinations of the 3 standard forms $\Xi_6[\delta_1]^2$, $F[\delta_1]$, and $F^c[\delta_1]$ given in Lemma 3. Q.E.D.

6.3.2 Orbits of pairs of admissible sextets in genus 3

We consider next the same issue of orbits and consistency of phase assignments for pairs of admissible sextets in genus 3. The following can be found by computer listings:

• The set of all pairs of asyzygous sextets may be decomposed into 7 mutually exclusive sets, according to whether the two sextets in the pair have 0, 1, 2, 3, 4, 5, or 6 spin structures in common. Each of these sets of pairs transforms transitively under the group of all modular transformations $Sp(6,\mathbf{Z})$. The number of pairs in each category is listed in the second column of the table below.

Also, we shall need the number of pairs of sextets, such that neither sextet in the pair contains a given spin structure Δ_1. The numbers of such pairs in each category is listed

in the third column of the table below. Under modular subgroup that preserves Δ_1, the sets with 0, 1, and 2 spin structures in common are NOT transitive. The table below lists in the fourth column the sizes of the orbits of the modular subgroup $Sp[\Delta_1](6,\mathbf{Z})$.

$\# \cap$	# pairs	# pairs $\not\supset \Delta_1$	Orbits	Reference pair
0	15120	10080	$5400^{(1)}$	$\{2,15,17,19,22,32\},\ \{9,10,13,16,27,28\}$
			$5400^{(2)}$	$\{4,5,10,16,26,36\},\ \{7,11,14,20,27,33\}$
1	30240	21000	840	$\{3,4,12,17,25,29\},\ \{10,20,22,25,28,35\}$
			$10080^{(1)}$	$\{5,15,22,27,29,31\},\ \{4,11,18,19,24,31\}$
			$10080^{(2)}$	$\{2,5,17,19,28,30\},\ \{2,8,10,15,25,32\}$
2	7560	5460	1260	$\{8,12,15,20,28,33\},\ \{4,12,13,17,24,33\}$
			1680	$\{4,5,10,16,26,36\},\ \{3,4,7,12,22,36\}$
			2520	$\{5,14,16,20,25,35\},\ \{6,16,24,25,30,36\}$
3	3360	2800	2800	$\{1,4,10,17,27,33\},\ \{4,10,11,17,25,26\}$
4	0	0	–	–
5	0	0	–	–
3	336	280	280	any pair

Table 1: Numbers of pairs of asyzygous sextets and modular orbits excluding Δ_1

• Consider the transformation law for sign assignments $\epsilon(\Delta;\{\Delta_i\},\{\Delta_i'\})$ for pairs of sextets in genus 3 given by the analogue of (6.22),

$$\epsilon(M\Delta;\{M\Delta_i\},\{M\Delta_i'\})\prod_{i=1}^{6}\epsilon^2(\Delta_i,M)\prod_{i=1}^{6}\epsilon^2(\Delta_i',M)=\epsilon(\Delta;\{\Delta_i\},\{\Delta_i'\}), \tag{6.31}$$

where M is any element of $Sp(6,\mathbf{Z})$. With computer calculations, using all the generators S, M_{A_i}, and M_{B_i}, $i=1,\cdots,6$ of the full $Sp(6,\mathbf{Z})$, the following may be shown.

1. A unique (up to a global sign) and consistent sign assignment exists for all the orbits in the sets with 0, 2 and 6 spin structures in common, as well as for the orbit $10080^{(2)}$ in the set with 1 spin structure in common;

2. No consistent sign assigment exists for any of the other orbits.

6.4 Branching rules for $Sp[\Delta](6,\mathbf{Z})$ orbits into $Sp[\delta](4,\mathbf{Z})$ orbits

In this section, we list the multiplicities of all the $Sp[\delta](4,\mathbf{Z})$ orbits which arise upon factorization of the orbits of $Sp[\Delta](6,\mathbf{Z})$. Recall that, in genus 3, the invariant set of pairs

of Δ-admissible sextets with p common spin structures can be decomposed further into irreducible orbits. Let these orbits be denoted by $\mathcal{Q}_{pq}$, with p indicating that the pairs of Δ-admissible sextets have p common spin structures, and q indicating which orbit is being considered for given p. In the table below, N_{pq} denotes the multiplicity of the genus 2 orbit in the decomposition of the orbit $\mathcal{Q}_{pq}$. Also, $\#(Q)$ denotes the cardinality of the genus 2 even spin structure orbit.

genus 2 orbit	$\#(Q)$	N_{01}	N_{02}	N_{21}	N_{22}	N_{23}	N_6
$Q_3^{0,0}$	6	12	0	0	0	0	0
$Q_3^{0,1}$	36	4	4	0	0	0	0
$Q_3^{1,1}$	18	0	4	0	0	0	0
$Q_4^{0,0}$	9	0	0	0	4	8	0
$Q_4^{0,1}$	18	0	0	8	0	0	0
$Q_4^{1,1}$	18	0	0	2	2	2	0
$Q_6^{0,0}$	6	0	6	6	3	0	6
$Q_6^{0,1}$	0	0	0	0	0	0	0
$Q_6^{1,1}$	9	2	0	2	0	2	4
Total number of pairs		234	252	234	90	126	72

Table 2: Branching rules for genus 3 orbits into genus 2 orbits

The computer analysis also shows that, in the above table, all copies of any given orbit $Q_p^{a,b}[\delta]$ always occur with the sign $+$. Thus there is no cancellation between the various copies of any orbit $Q_p^{a,b}[\delta]$. (Of course, the global sign in front of each $F_p^{a,b}[\delta]$ is a matter of convention, depending on the choice of global sign for the definition of $F_p^{a,b}[\delta]$.

To each $Sp[\Delta](6,\mathbf{Z})$ orbit $\mathcal{Q}_{pq}$, we can associate then a polynomial P_{pq} in genus 2 ϑ-constants, defined as the linear combination of the polynomials $F_p^{a,b}[\delta]$, with coefficients given by the multiplicities with which the $Sp[\delta](4,\mathbf{Z})$ orbit $Q_p^{a,b}[\delta]$ appears. Thus P_{01} and P_{02} stand for the two polynomials corresponding to the two genus 3 orbits of pairs with 0 common spin structures; P_{21}, P_{22}, P_{23} stand for the 3 orbits of pairs with 2 common spin structures; and P_6 stands for the single orbit of pairs with 6 common spin structures. The overall sign of each polynomial is arbitrary. The relative signs are of course fixed by the stabilizer group of the genus 3 spin structure Δ. We have (we omit reference to Δ in F),

$$\begin{aligned}
P_{01} &= -12F_3^{0,0} + 4F_3^{0,1} + 2F_6^{1,1} && = -12\Xi_6^2 + 6F - 2F^c \\
P_{02} &= -4F_3^{0,1} + 4F_3^{1,1} + 6F_6^{0,0} && = 6F + 6F^c \\
P_{21} &= +8F_4^{0,1} - 2F_4^{1,1} + 6F_6^{0,0} + 2F_6^{1,1} && = 6F + 12F^c
\end{aligned}$$

$$\begin{array}{rcll} P_{22} &=& -4F_4^{0,0} - 2F_4^{1,1} + 3F_6^{0,0} & = 4\Xi_6^2 + 3F + 2F^c \\ P_{23} &=& -8F_4^{0,0} - 2F_4^{1,1} + 2F_6^{1,1} & = 8\Xi_6^2 + 4F^c \\ P_6 &=& +6F_6^{0,0} + 4F_6^{1,1} & = 6F + 4F^c \end{array} \tag{6.32}$$

where we have used (6.24) to express all of these in terms of the quantities Ξ_6^2, F and F^c. The previous discussion results in the following lemma:

Lemma 4. *Let the sign assignments $\epsilon(\Delta; \{\Delta_i\}, \{\Delta_i'\})$ satisfy the transformation (6.31) for each orbit $\mathcal{Q}_{pq}$. Then we have*

$$\lim_{t\to 0} \sum_{(\{\Delta_i\},\{\Delta_i'\})\in\mathcal{Q}_{pq}} \epsilon(\Delta; \{\Delta_i\}, \{\Delta_i'\}) \prod_{i=1}^{6} \vartheta[\Delta_i](0,\Omega^{(3)})^2 \vartheta[\Delta_i'](0,\Omega^{(3)})^2 = 2^8 \eta(\Omega^{(1)})^{24} P_{pq}(\Omega^{(2)}) \tag{6.33}$$

6.5 Candidates for $\Xi_6[\Delta](\Omega^{(3)})^2$

Each orbit $\mathcal{Q}_{pq}$ contributes a consistent term to the candidate for the genus 3 superstring measure, transforming covariantly under $Sp(6,\mathbf{Z})$ transformations. Thus we can take an arbitrary linear combination of these orbits and obtain a modular covariant expression

$$\sum_{p,q} N_{pq} \sum_{(\{\Delta_i\},\{\Delta_i'\})\in\mathcal{Q}_{pq}} \epsilon(\Delta; \{\Delta_i\}, \{\Delta_i'\}) \prod_{i=1}^{6} \vartheta[\Delta_i](0,\Omega^{(3)})^2 \vartheta[\Delta_i'](0,\Omega^{(3)})^2 \tag{6.34}$$

Candidates for $\Xi_6[\Delta](\Omega^{(3)})^2$ must tend to $2^8\eta(\Omega^{(1)})^2\,\Xi_6[\delta](\Omega^{(2)})^2$. In view of Lemma 4, the limit at $t \to 0$ of the linear combination (6.34) will be a multiple of $\eta(\Omega^{(1)})^2\,\Xi_6[\delta](\Omega^{(2)})^2$ if the multiplicities N_{pq} satisfy

$$\begin{array}{rcl} 2N_{01} + 2N_{02} + 2N_{21} + N_{22} + 2N_6 &=& 0 \\ -N_{01} + 3N_{02} + 6N_{21} + N_{22} + 2N_{23} + 2N_6 &=& 0 \end{array} \tag{6.35}$$

in which case the limit is given by

$$2^8\ \eta(\Omega^{(1)})^2\ (-12\,N_{01} + 4\,N_{22} + 8\,N_{23})\ \Xi_6[\delta](\Omega^{(2)})^2 \tag{6.36}$$

It is convenient to summarize our findings in the following theorem:

Theorem 2. *Let the genus 3 expression $\Xi_6[\Delta](\Omega^{(3)})^2$ be defined by (**??**), where Q_{pq} are the orbits of pairs of Δ-admissible sextets from Table. Assume that the multiplicities N_{pq} satisfy the condition (6.35), and set $N = -12\,N_{01} + 4\,N_{22} + 8\,N_{23}$. Let Δ factorize into*

an even spin structure δ at genus 2. Then the expression $\Xi_6[\Delta](\Omega^{(3)})^2$ satisfies the three conditions

(*i'*) $\Xi_6[\Delta](\Omega^{(3)})^2$ *is holomorphic on the Siegel upper half space*;

(*ii'*) $\Xi_6[\tilde\Delta](\tilde\Omega^{(3)})^2 = \det(C\Omega^{(3)}+D)^{12}\,\Xi_6[\Delta](\Omega^{(3)})^2$;

(*iii'*) $\lim_{t\to 0}\Xi_6[\Delta](\Omega^{(3)})^2 = \eta(\Omega^{(1)})^{24}\Xi_6[\delta](\Omega^{(2)})^2$.

For example, an integer combination leading to a multiple of $2^8\,\eta(\Omega^{(1)})^2\Xi_6[\delta](\Omega^{(2)})^2$ by the square of an integer is $N_{01}=-2$, $N_{02}=4$, $N_{21}=-2$, $N_{23}=-1$, in which case we get

$$\begin{aligned}\lim_{t\to 0}\sum_{(\{\Delta_i\},\{\Delta_i'\})\in\mathcal{Q}_{pq}}&\epsilon(\Delta;\{\Delta_i\},\{\Delta_i'\})\prod_{i=1}^{6}\vartheta[\Delta_i](0,\Omega^{(3)})^2\vartheta[\Delta_i'](0,\Omega^{(3)})^2\\ &=16\cdot 2^8\eta(\Omega^{(1)})^{24}\;\Xi_6[\delta](\Omega^{(2)})^2.\end{aligned}\tag{6.37}$$

6.6 Vanishing of the genus 3 cosmological constant

We address a final issue of physical and mathematical significance, namely the behavior of the genus 3 *cosmological constant*, defined by

$$\Upsilon_8\equiv\sum_\Delta\Xi_6[\Delta](\Omega^{(3)})\vartheta[\Delta](0,\Omega^{(3)})^4\tag{6.38}$$

By the its very construction, $\Xi_6[\Delta]$ transforms under the modular group $Sp(6,\mathbf{Z})$ as $\vartheta[\Delta](0,\Omega^{(3)})^{12}$, and therefore the quantity Υ_8 is a genus 3 modular form of weight 8. An infinite family of modular forms of weight $4k$ may be generated as follows,

$$\Psi_{4k}(\Omega^{(3)})\equiv\sum_\Delta\vartheta[\Delta](0,\Omega^{(3)})^{8k}\tag{6.39}$$

for k any positive integer. In [14], it was argued that $\Psi_8=\Psi_4^2/8$, based on asymptotic identifications and numerical calculations. We shall assume that this is the only independent holomorphic modular form of weight 8, as we are not aware of any proof that this statement is true. Given this assumption, as well as the asymptotic behavior established in this paper for $\Xi_6[\Delta](\Omega^{(3)})$, as the surface undergoes a separating degeneration, it is clear that the modular form Υ_8 must vanish in this limit. But Ψ_8 is non-zero in the same limit. As a result, $\Upsilon_8=0$ throughout moduli space, and the cosmological constant vanishes to three loop order.

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